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ORIGINAL PAPER

A hybrid type four-step method with vanished phase-lag and its first, second and third derivatives for each level for the numerical integration of the Schrödinger equation

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Abstract In this paper, we study the effects of the vanishing of the phase-lag and its first, second and third derivatives on the effectiveness of a four-step hybrid type method of sixth algebraic order. As a result of the above described study, a Hybrid type of three level four-step method of sixth algebraic order is obtained. We investigate the new produced method theoretically and computationally. The theoretical investigation of the new hybrid method consists of:

- The development of the new method.
- The computation of the Local Truncation Error.
- The Comparison of the Local Truncation Error analysis with other known methods of the same form.
- The Stability Analysis.

The computational investigation consists of the application of the new obtained hybrid method to the numerical solution of the resonance problem of the radial time independent Schrödinger equation.

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1 Introduction

The subject of the present paper is the investigation of the efficient solution of the boundary value problems of the form of the radial time independent Schrödinger equation:

$$p''(r) = [l(l+1)/r^2 + V(r) - k^2]\, p(r), \tag{1}$$

In the above mentioned mathematical model, we have the following:

- The function $W(r) = l(l+1)/r^2 + V(r)$ is called *the effective potential*. This function satisfies the following relation: $W(x) \to 0$ as $x \to \infty$.
- The quantity k^2 is a real number denoting *the energy*,
- The quantity l is a given integer representing the *angular momentum*, and finally
- V is a given function denotes the *potential*.

For the above mentioned boundary value problem, we have the following boundary conditions:

$$p(0) = 0 \tag{2}$$

and a second boundary condition, for large values of r, determined by physical considerations.

The problems with mathematical models of the form of the radial Schrödinger equation belong to special second-order initial or boundary value problems of the form:

$$p''(r) = f(r, p(r)), \tag{3}$$

with solutions which have periodical and/or oscillatory behavior.

This category of problems has a main characteristic. The characteristic is that their mathematical model is described by a system of second order ordinary differential equations of the form (3) in which the first derivative p' does not appear explicitly. Mathematical models with the above main characteristic can be found in many problems of applied sciences (i.e., astronomy, astrophysics, quantum mechanics, quantum chemistry, celestial mechanics, electronics, physical chemistry, chemical physics,. . ., etc) (see for example [1–4]).

For the numerical solution of the above mentioned problems, much research has been done in the last decades. The development of effective , fast and reliable algorithms is the target of this research (see for example [5–105]). The main categories of finite difference methods on which important developments on research and innovation was taken place during the last decades are shown in Fig. 1.

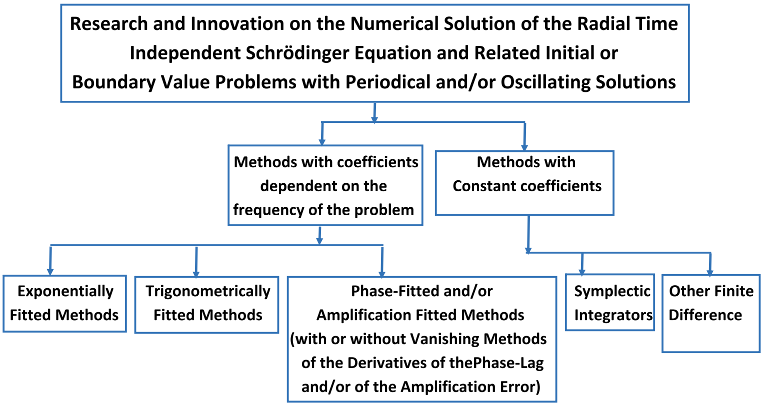


Fig. 1 Categories of the finite difference methods developed the last decades

Some bibliography on the above mentioned research is given below :

- Phase-fitted methods and numerical methods with minimal phase-lag of Runge–Kutta and Runge–Kutta–Nyström type have been obtained in [5–11].
- In [12–17], exponentially and trigonometrically fitted Runge–Kutta and Runge–Kutta–Nyström methods are constructed.
- Multistep phase-fitted methods and multistep methods with minimal phase-lag are obtained in [22–48].
- Symplectic integrators are investigated in [49–77].
- Exponentially and trigonometrically multistep methods have been produced in [78–98].
- Nonlinear methods have been studied in [99] and [100].
- Review papers have been presented in [101–105].
- Special issues and Symposia in International Conferences have been developed on this subject (see [106–110]).

We will study a new approach to develop numerical methods for the problems of the form (3) with periodical and/or oscillating behavior of the solution. This approach is based on the *Hybrid* form of the proposed family of methods, i.e. is based on the fact that the proposed family of methods has more than one stage. The new approach is based on the idea of vanishing of the phase-lag and its derivatives for each lever (or block of levels) of the hybrid method. Our study will also investigate how this vanishing of the phase-lag and its derivatives for each lever (or block of levels) of the hybrid method affects the effectiveness of the produced hybrid method. Finally, we will study if the obtained hybrid method is more efficient than other numerical methods which have vanished phase-lag and its derivatives for the whole of methods and not in every stage or block of stages.

Remark 1 The numerical methods obtained with the above described approach are very efficient on any problem with periodic and/or oscillating solution or solutions which contains the functions cos and sin or a combination of them.

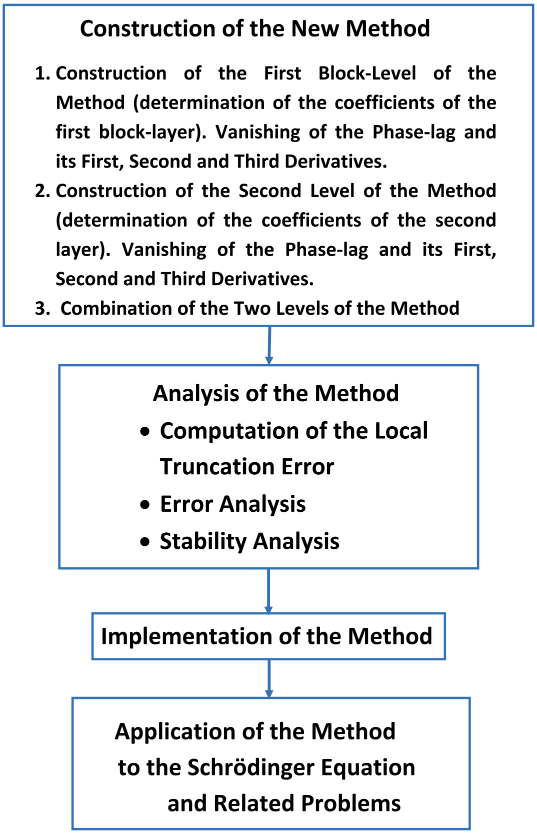


Fig. 2 Flowchart of the presentation of the analysis of the new proposed Hybrid type method

The main scopes of this paper are the following:

- The determination of the coefficients of the proposed Hybrid type three-stage four-step method in order to have
  1. high algebraic order,
  2. vanished phase-lag on each stage or block of stages of the method,
  3. vanished the first derivative of the phase-lag on each stage or block of stages of the method,
  4. vanished the second derivative of the phase-lag on each stage or block of stages of the method,
  5. vanished the third derivative of the phase-lag on each stage or block of stages of the method.
- The investigation of the local truncation error. We will compare the local truncation error of the new obtained hybrid type method with other methods of the same form (comparative local truncation error analysis.
- The investigation of the stability.
- The investigation of the efficiency of the new proposed method by applying it to the numerical solution of the resonance problem of the radial time independent

Schrödinger equation. This is one of the most difficult problems arising from the one-dimensional Schrödinger equation.

In [25] and [28] Simos and his coworkers have developed direct formula for the computation of the phase-lag for any $2\,m$-method symmetric multistep method. Based on this formula, we will also calculate the phase-lag's first, second and third derivatives.

In Fig. 2, we present the flowchart of the presentation of the analysis of the new proposed Runge–Kuttta type method.

The phase-lag analysis of symmetric $2\,n$-methods is presented in Sect. 2. In Sect. 3, we present the construction of the new hybrid type three-stage four-step method. The investigation of the local truncation error (LTE) of the developed method and the comparison with the LTE of other well known methods of the same type, is presented in Sect. 4. In Sect. 5, we present the stability analysis with frequency of the scalar test equation of the stability analysis not equal with the frequency of the scalar test equation of the phase-lag analysis. Numerical results are presented in Sect. 6. Finally, remarks and conclusions are shown in Sect. 7.

2 Phase-lag analysis of symmetric $2\,n$-step methods

In order to solve numerically the initial value problem

$$p'' = f(x, p), \tag{4}$$

one may use a multistep method with q steps. We divide the interval $[a, b]$ into equally spaced intervals using $\{x_i\}_{i=0}^{p} \subset [a, b]$ and $h = |x_{i+1} - x_i|$, $i = 0(1)p - 1$.

We study the case of symmetric multistep methods, i.e.,

$$a_i = a_{p-i}, \quad b_i = b_{p-i}, \quad i = 0(1)\frac{p}{2}. \tag{5}$$

Application of a symmetric $2\,q$-step method, that is for $i = -q(1)q$, to the scalar test equation

$$p'' = -w^2\, p, \tag{6}$$

leads to the following difference equation

$$A_q(v)\, p_{n+q} + \cdots + A_1(v)\, p_{n+1} + A_0(v)\, p_n + A_1(v)\, p_{n-1} + \cdots + A_q(v) p_{n-q} = 0, \tag{7}$$

where $v = w\,h$, h is the step length and $A_0(v)$, $A_1(v), \ldots, A_q(v)$ are polynomials.

The characteristic equation [which is associated with (7)] is given by:

$$A_q(v)\,\lambda^q + \cdots + A_1(v)\quad \lambda + A_0(v) + A_1(v)\quad \lambda^{-1} + \cdots + A_q(v)\quad \lambda^{-q} = 0 \tag{8}$$

Theorem 1 [25,28] *The symmetric $2\,q$-step method with characteristic equation given by (8) has phase-lag order k and phase-lag constant c given by:*

$$- c\, v^{k+2} + O(v^{k+4}) = \frac{2\, A_q(v)\, \cos(q\, v) + \cdots + 2\, A_j(v)\, \cos(j\, v) + \cdots + A_0(v)}{2\, q^2\, A_q(v) + \cdots + 2\, j^2\, A_j(v) + \cdots + 2\, A_1(v)} \tag{9}$$

Remark 2 The formula (9) is a direct method for the calculation of the phase-lag of any symmetric $2\,q$-step method.

Remark 3 For the method studied in this paper—for the hybrid type symmetric four-step method—the number $q = 2$ and the direct formula for the computation of the phase-lag is given by:

$$-c\, v^{k+2} + O(v^{k+4}) = \frac{2\, A_2(v)\, \cos(2\, v) + 2\, A_1(v)\, \cos(v) + A_0(v)}{8\, A_2(v) + 2\, A_1(v)} \tag{10}$$

where k is the phase-lag order and c is the phase-lag constant.

3 The new proposed method

Let us consider the family of hybrid type symmetric four-step methods for the numerical solution of problems of the form $p'' = f(x, p)$:

$$\begin{aligned} \bar{p}_n &= p_n - a_0\, h^2\, \left(p''_{n+1} - 2\, p''_n + p''_{n-1}\right) \\ \bar{p}_{n+2} &= -a_1\, p_{n+1} - 2\, p_n - a_1\, p_{n-1} - p_{n-2} \\ &\quad + h^2 \left(b_0\, p''_{n+1} + b_1\, \bar{p}''_n + b_0\, p''_{n-1}\right) \\ & p_{n+2} + a_2\, p_{n+1} + 2\, p_n + a_2\, p_{n-1} + p_{n-2} \\ &= h^2\, \left[b_4\, \left(\bar{p}''_{n+2} + p''_{n-2}\right) + b_3\, \left(p''_{n+1} + p''_{n-1}\right) + b_2\, p''_n\right], \end{aligned} \tag{11}$$

where the coefficient $a_j,\ j = 0, 1$ and $b_i,\ i = 0(1)4$ are free parameters, h is the step size of the integration , n is the number of steps, p_n is the approximation of the solution on the point x_n, $x_n = x_0 + n\, h$ and x_0 is the initial value point.

In the flowchart of Fig. 3, we present the development of the new proposed method.

3.1 First block level of the hybrid method

Based on the above flowchart, we consider the first block level which consists of the two first stages of the above mentioned hybrid type method:

$$\begin{aligned} &\bar{p}_n = p_n - a_0\, h^2\, \left(p''_{n+1} - 2\, p''_n + p''_{n-1}\right) \\ &p_{n+2} + a_1\, p_{n+1} + 2\, p_n + a_1\, p_{n-1} + p_{n-2} = h^2 \left(b_0\, p''_{n+1} + b_1\, \bar{p}''_n + b_0\, p''_{n-1}\right) \end{aligned} \tag{12}$$

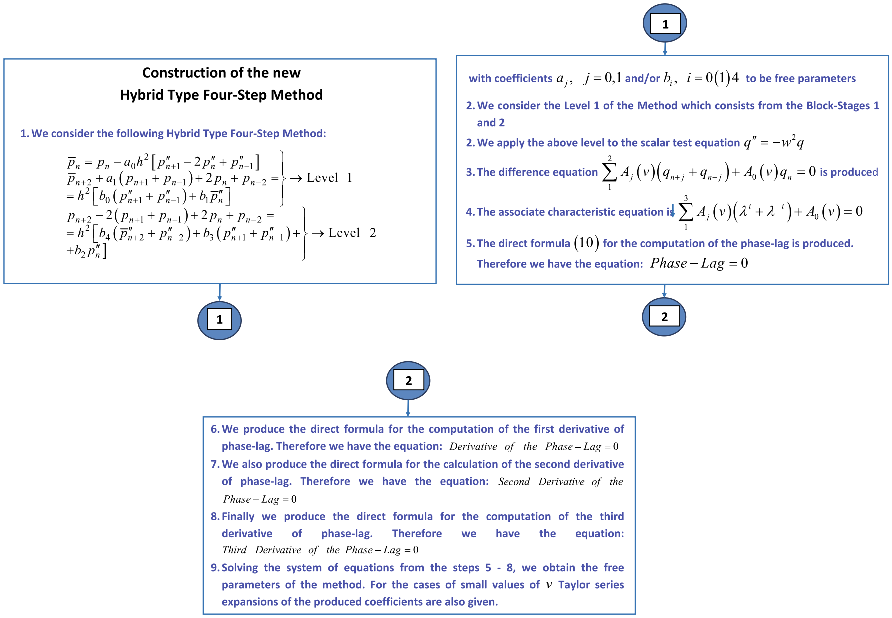


Fig. 3 Flowchart of the construction of any method of the family

Applying the above block level to the scalar test Eq. (6), we get the difference Eq. (7) with $q = 2$ and $A_j(v),\ \ j = 0,\ 1,\ 2$ given by:

$$A_2(v) = 1,\ A_1(v) = a_1 + v^2 \left(v^2\, a_0\, b_1 + b_0\right)$$
$$A_0(v) = 2 + v^2\, b_1 - 2\, v^4\, a_0\, b_1 \tag{13}$$

We require the above first block level of the method (11) to have vanished the phase-lag and its first, second and third derivatives. Therefore, we obtain the following system of equations [using the formula (10) and (13)]:

$$\text{Phase-Lag} = \frac{1}{2}\, \frac{F_1}{v^4 a_0 b_1 + v^2 b_0 + a_1 + 4} = 0 \tag{14}$$

$$\text{First Derivative of the Phase-Lag} = -\frac{F_2}{\left(v^4\, a_0\, b_1 + v^2\, b_0 + a_1 + 4\right)^2} = 0 \tag{15}$$

$$\text{Second Derivative of the Phase-Lag} = -\frac{F_3}{\left(v^4\, a_0\, b_1 + v^2\, b_0 + a_1 + 4\right)^3} = 0 \tag{16}$$

$$\text{Third Derivative of the Phase-Lag} = \frac{F_4}{\left(v^4\, a_0\, b_1 + v^2\, b_0 + a_1 + 4\right)^4} = 0 \tag{17}$$

where

$$
\begin{aligned}
F_1 =\; & 2\cos(v)v^4 a_0 b_1 - 2 v^4 a_0 b_1 + 2\cos(v) v^2 b_0 \\
& + v^2 b_1 + 2\cos(v) a_1 + 4(\cos(v))^2 \\
F_2 =\; & \sin(v) v^8 a_0{}^2 b_1{}^2 + 2\sin(v) v^6 a_0 b_0 b_1 \\
& + 4\cos(v)\sin(v) v^4 a_0 b_1 + 2\sin(v) v^4 a_0 a_1 b_1 \\
& + 2 v^5 a_0 b_0 b_1 + v^5 a_0 b_1{}^2 + 8(\cos(v))^2 v^3 a_0 b_1 \\
& + 4\sin(v) v^4 a_0 b_1 + \sin(v) v^4 b_0{}^2 \\
& - 16\cos(v) v^3 a_0 b_1 + 4 v^3 a_0 a_1 b_1 \\
& + 4\cos(v)\sin(v) v^2 b_0 + 2\sin(v) v^2 a_1 b_0 \\
& + 16 v^3 a_0 b_1 + 4(\cos(v))^2 v b_0 + 4\sin(v) v^2 b_0 \\
& + 4\cos(v)\sin(v) a_1 - 8\cos(v) v b_0 + \sin(v) a_1{}^2 \\
& - v a_1 b_1 + 16\cos(v)\sin(v) + 4\sin(v) a_1 - 4 v b_1 \\
F_3 =\; & -64 + \cos(v) v^6 b_0{}^3 + 8(\cos(v))^2 v^4 b_0{}^2 \\
& - 12(\cos(v))^2 v^2 b_0{}^2 + 16\sin(v) v^3 b_0{}^2 \\
& + 64(\cos(v))^2 v^2 b_0 + 24\cos(v) v^2 b_0{}^2 \\
& - 8\cos(v) a_1 b_0 + 4(\cos(v))^2 a_1 b_0 \\
& - 4 v^8 a_0{}^2 b_1{}^2 - 8 v^2 a_1 b_0 - 32\cos(v)\sin(v) \\
& \quad v^3 a_0 a_1 b_1 + 6\cos(v) v^6 a_0 a_1 b_0 b_1 \\
& - 48\cos(v)\sin(v) v^5 a_0 b_0 b_1 + 16\sin(v) \\
& \quad v a_1 b_0 - 6 v^8 a_0{}^2 b_0 b_1{}^2 + 8\cos(v) v^4 b_0{}^2 \\
& - 3 v^8 a_0{}^2 b_1{}^3 - 80 v^6 a_0{}^2 b_1{}^2 + 12 v^2 b_0 b_1 - 4 a_1{}^2 \\
& + 3\cos(v) v^2 a_1{}^2 b_0 + 16(\cos(v))^2 v^2 a_1 b_0 \\
& - 20 v^6 a_0{}^2 a_1 b_1{}^2 + 2 v^6 a_0 b_0{}^2 b_1 + v^6 a_0 b_0 b_1{}^2 \\
& + 12 v^4 a_0 a_1 b_1{}^2 + 12 v^2 a_0 a_1{}^2 b_1 + 3 v^2 a_1 b_0 b_1 \\
& + \cos(v) v^{12} a_0{}^3 b_1{}^3 + 8(\cos(v))^2 v^8 \\
& \quad a_0{}^2 b_1{}^2 + 8\cos(v) v^8 a_0{}^2 b_1{}^2 - 40(\cos(v))^2 \\
& \quad v^6 a_0{}^2 b_1{}^2 + 32\sin(v) v^7 a_0{}^2 b_1{}^2 + 80\cos(v) v^6 \\
& \quad a_0{}^2 b_1{}^2 + 64(\cos(v))^2 v^4 a_0 b_1 \\
& + 3\cos(v) v^4 a_1 b_0{}^2 - 16\cos(v)\sin(v) v^3 b_0{}^2 \\
& + 96(\cos(v))^2 v^2 a_0 b_1 + 16\cos(v) v^4 a_0 b_1 \\
& - 8 v^6 a_0 b_0 b_1 - 8 v^4 a_0 a_1 b_1 + 24 v^4 a_0 b_0 b_1 \\
& + 96 v^2 a_0 a_1 b_1 + 16\cos(v) v^2 b_0 - 32 v^4 a_0 b_1 \\
& + 48 v^4 a_0 b_1{}^2 + 192 v^2 a_0 b_1 + 128(\cos(v))^2 \\
& - 32 a_1 - 16 b_1 - 16\cos(v)\sin(v) v a_1 b_0 \\
& + 32\sin(v) v^3 a_0 a_1 b_1 - 48\cos(v) v^2 a_0 a_1 b_1 \\
& - 128\cos(v)\sin(v) v^3 a_0 b_1 + 24(\cos(v))^2
\end{aligned}
$$

$$v^2\, a_0\, a_1\, b_1 + 16\, \cos(v)\, v^4\, a_0\, a_1\, b_1 + 72\, \cos(v)$$
$$v^4\, a_0\, b_0\, b_1 + 48\, \sin(v)\, v^5\, a_0\, b_0\, b_1$$
$$+ 3\, \cos(v)\, v^4\, a_0\, {a_1}^2\, b_1 + 16\, (\cos(v))^2\, v^4$$
$$a_0\, a_1\, b_1 - 36\, (\cos(v))^2\, v^4\, a_0\, b_0\, b_1$$
$$+ 16\, (\cos(v))^2\, v^6\, a_0\, b_0\, b_1 + 16\, \cos(v)\, v^6$$
$$a_0\, b_0\, b_1 - 32\, \cos(v)\, \sin(v)\, v^7\, {a_0}^2\, {b_1}^2$$
$$+ 3\, \cos(v)\, v^8\, a_0\, {b_0}^2\, b_1 + 3\, \cos(v)\, v^{10}\, {a_0}^2\, b_0\, {b_1}^2$$
$$+ 3\, \cos(v)\, v^8\, {a_0}^2\, a_1\, {b_1}^2 + 6\, v^4\, a_0\, a_1\, b_0\, b_1$$
$$- 64\, \cos(v)\, \sin(v)\, v\, b_0 + 16\, \cos(v)\, v^2\, a_1\, b_0$$
$$+ 128\, \sin(v)\, v^3\, a_0\, b_1 - 192\, \cos(v)\, v^2\, a_0\, b_1 + 64\, \sin(v)$$
$$v\, b_0 - 32\, v^2\, b_0 - 8\, a_1\, b_1 - 32\, \cos(v)\, b_0 + 64\, (\cos(v))^2\, a_1$$
$$+ 16\, (\cos(v))^2\, b_0 + 8\, \cos(v)\, {a_1}^2 - 4\, v^4\, {b_0}^2$$
$$- {a_1}^2\, b_1 + 8\, (\cos(v))^2\, {a_1}^2 + \cos(v)\, {a_1}^3 + 16\, \cos(v)\, a_1$$

$$F_4 = 96\, \sin(v)\, v^6\, {a_0}^2\, a_1\, {b_1}^2 - 192\, (\cos(v))^2$$
$$v^5\, a_0\, {b_0}^2\, b_1 - 960\, \cos(v)\, v^5\, {a_0}^2\, a_1\, {b_1}^2$$
$$+ 288\, \sin(v)\, v^6\, a_0\, {b_0}^2\, b_1 + 1152\, (\cos(v))^2\, v^5$$
$$a_0\, b_0\, b_1 + \sin(v)\, v^8\, {b_0}^4 + 48\, (\cos(v))^2\, v^5\, {b_0}^3$$
$$+ 12\, \sin(v)\, v^6\, {b_0}^3 - 24\, \cos(v)\, v^5\, {b_0}^3 + 72\, \sin(v)$$
$$v^4\, {b_0}^3 - 48\, (\cos(v))^2\, v^3\, {b_0}^3 + 384\, (\cos(v))^2$$
$$v^3\, {b_0}^2 + 96\, \cos(v)\, v^3\, {b_0}^3 + 16\, \cos(v)\, \sin(v)\, {a_1}^3$$
$$+ 192\, (\cos(v))^2\, v\, {b_0}^2 - 1536\, v\, a_0\, b_1 - 192\, v\, a_1\, b_0$$
$$+ 192\, \sin(v)\, v^2\, {b_0}^2 - 384\, \cos(v)\, v\, b_0 + 192\, \cos(v)$$
$$\sin(v)\, {a_1}^2 - 48\, v^{11}\, {a_0}^3\, {b_1}^3 + 64\, \sin(v)\, v^2\, b_0$$
$$- 384\, \cos(v)\, v\, {b_0}^2 - 768\, v^3\, a_0\, b_1 + 768\, (\cos(v))^2\, v\, b_0$$
$$+ 48\, \sin(v)\, v^4\, {b_0}^2 + 768\, \cos(v)\, \sin(v)\, a_1$$
$$+ 384\, \cos(v)\, \sin(v)\, b_0 - 48\, v^3\, a_1\, {b_0}^2 - 24\, \sin(v)\, {a_1}^2\, b_0$$
$$- 192\, \sin(v)\, a_1\, b_0 - 12\, v^{11}\, {a_0}^3\, {b_1}^4 - 480\, v^9\, {a_0}^3\, {b_1}^3$$
$$+ 6\, \sin(v)\, v^{12}\, {a_0}^2\, {b_0}^2\, {b_1}^2 - 240\, \cos(v)\, \sin(v)$$
$$v^{10}\, {a_0}^3\, {b_1}^3 + 240\, (\cos(v))^2\, v^9\, {a_0}^2\, b_0\, {b_1}^2$$
$$+ 36\, \sin(v)\, v^{10}\, {a_0}^2\, b_0\, {b_1}^2 - 24\, v^{11}\, {a_0}^3\, b_0\, {b_1}^3$$
$$+ 120\, v^7\, {a_0}^2\, a_1\, {b_1}^3 + 1152\, (\cos(v))^2\, v^3\, a_0\, b_0 b_1$$
$$+ 48\, (\cos(v))^2\, v\, {a_1}^2\, b_0 + 192\, \cos(v)\, \sin(v)$$
$$v^4\, {b_0}^2 + 48\, \cos(v)\, \sin(v)\, v^2\, {a_1}^2\, b_0 + 96\, \sin(v)$$
$$v^6\, a_0\, b_0\, b_1 - 96\, v\, a_1\, b_0\, b_1 - 288\, v\, a_0\, {a_1}^2\, b_1 + 768\, \cos(v)\, \sin(v)$$
$$v^4\, a_0\, b_1 + 96\, \sin(v)\, v^4\, a_0\, a_1\, b_1 - 48\, \cos(v)\, \sin(v)$$
$$v^2\, a_1\, {b_0}^2 - 24\, v\, {a_1}^2\, b_0 + 12\, \sin(v)\, v^{12}\, {a_0}^3\, {b_1}^3$$
$$+ 144\, \cos(v)\, \sin(v)\, v^2\, a_0\, {a_1}^2\, b_1 + 48\, \sin(v)\, v^4\, a_0\, a_1\, b_0\, b_1$$

$$
\begin{aligned}
&- 240\,(\cos(v))^2\,v^9\,a_0{}^3\,b_1{}^3 + 48\,\cos(v)\,\sin(v)\\
&\quad v^8\,a_0\,b_0{}^2\,b_1 + 192\,v^5\,a_0\,b_0\,b_1{}^2 - 60\,v^3\,a_0\,a_1{}^2\,b_1{}^2\\
&- 144\,\cos(v)\,v^5\,a_0\,a_1\,b_0\,b_1 + 96\,\cos(v)\,v\,a_0\,a_1{}^2\,b_1\\
&- 1152\,\sin(v)\,v^2\,a_0\,a_1\,b_1 + 768\,\cos(v)\,v\,a_0\,a_1\,b_1\\
&+ 96\,v^7\,a_0{}^2\,a_1\,b_0\,b_1{}^2 + 48\,v^5\,a_0\,a_1\,b_0\,b_1{}^2 + 24\,v^3\,a_0\,a_1{}^2\,b_0\,b_1\\
&+ 4\,\sin(v)\,v^{14}\,a_0{}^3\,b_0\,b_1{}^3 + 96\,(\cos(v))^2\,v^{11}\,a_0{}^3\,b_1{}^3\\
&+ 16\,\cos(v)\,\sin(v)\,v^{12}\,a_0{}^3\,b_1{}^3 + 4\,\sin(v)\,v^{12}\,a_0{}^3\,a_1\,b_1{}^3\\
&- 48\,\cos(v)\,v^3\,a_0\,a_1{}^2\,b_1 - 480\,v^3\,a_0\,a_1\,b_1{}^2 - 120\,v^9\,a_0{}^2\,b_0\,b_1{}^2\\
&+ 96\,(\cos(v))^2\,v^3\,a_1\,b_0{}^2 + 6\,\sin(v)\,v^4\,a_1{}^2\,b_0{}^2\\
&- 72\,\cos(v)\,\sin(v)v^4\,b_0{}^3 - 3840\,\cos(v)\,v^5\,a_0{}^2\,b_1{}^2\\
&- 768\,\cos(v)\,v^3\,a_0\,b_1 - 1152\,v\,a_0\,a_1\,b_1 + 384\,v^3\,a_0\,b_0\,b_1\\
&+ 64\,\sin(v)\,v^4\,a_0\,b_1 - 576\,v^5\,a_0\,b_0\,b_1 - 384\,v^3\,a_0\,a_1\,b_1\\
&+ 48\,\sin(v)\,v^8\,a_0{}^2\,b_1{}^2 + 1536\,(\cos(v))^2\,v^3\,a_0\\
&\quad b_1 + 768\,\cos(v)\,\sin(v)\,v^2\,b_0 + 96\,\sin(v)\,v^2\,a_1\,b_0\\
&+ 12\,v^9\,a_0{}^2\,b_0\,b_1{}^3 + 48\,\sin(v)\,v^2\,a_1\,b_0{}^2\\
&+ 192\,\sin(v)\,v^4\,a_0\,b_0\,b_1 - 24\,v^5\,b_0{}^3 + 1152\,\cos(v)\,\sin(v)\\
&\quad v^2\,a_0\,a_1\,b_1 + 4\,\sin(v)\,v^6\,a_1\,b_0{}^3 + 384\,\sin(v)\,v^6\\
&\quad a_0{}^2\,b_1{}^2 + 768\,(\cos(v))^2\,v^3\,a_0\,a_1\,b_1 - 192\,\cos(v)\\
&\quad v\,a_1\,b_0 - 384\,v\,b_0 + 48\,\sin(v)\,a_1{}^2 + 64\,\sin(v)\,a_1\\
&- 144\,v^5\,a_0\,a_1\,b_0\,b_1 - 288\,(\cos(v))^2\,v^7\,a_0{}^2\,b_0\\
&\quad b_1{}^2 - 2304\,\cos(v)\,v^3\,a_0\,b_0\,b_1 + 192\,\cos(v)\,\sin(v)\,v^8\\
&\quad a_0{}^2\,b_1{}^2 + 36\,\sin(v)\,v^8\,a_0{}^2\,a_1\,b_1{}^2 - 144\,\sin(v)\,v^2\\
&\quad a_0\,a_1{}^2\,b_1 + 456\,\sin(v)\,v^8\,a_0{}^2\,b_0\,b_1{}^2\\
&- 384\,(\cos(v))^2 v\,a_0\,a_1\,b_1 + 192\,(\cos(v))^2\,v^7\,a_0\\
&\quad b_0{}^2\,b_1 - 96\,\cos(v)\,v^7\,a_0{}^2\,a_1\,b_1{}^2 + 2304\,\cos(v)\,\sin(v)\\
&\quad v^2\,a_0\,b_1 + 576\,\cos(v)\,v^7\,a_0{}^2\,b_0\,b_1{}^2 + 36\,\sin(v)\,v^8\,a_0\\
&\quad b_0{}^2\,b_1 + 480\,(\cos(v))^2\,v^5\,a_0{}^2\,a_1\,b_1{}^2 + 384\,\cos(v)\\
&\quad \sin(v)\,v^2\,a_1\,b_0 - 384\,\cos(v)\,\sin(v)\,v^6\,a_0{}^2\,b_1{}^2\\
&- 96\,\cos(v)\,v^7\,a_0\,b_0{}^2\,b_1 + 6\,\sin(v)\,v^8\,a_0{}^2\,a_1{}^2\,b_1{}^2\\
&+ 1024\,\cos(v)\,\sin(v) - 48\,(\cos(v))^2\,v\,a_0\,a_1{}^2\,b_1\\
&+ 12\,v^3\,a_1\,b_0{}^2\,b_1 + 36\,\sin(v)\,v^4\,a_1\,b_0{}^2 - 96\,\cos(v)\\
&\quad \sin(v)\,v^6\,a_0{}^2\,a_1\,b_1{}^2 + 384\,\cos(v)\,\sin(v)\,v^4\,a_0\\
&\quad a_1\,b_1 + 288\,(\cos(v))^2\,v^3\,a_0\,a_1\,b_0\,b_1\\
&+ 1920\,(\cos(v))^2\,v^5\,a_0{}^2\,b_1{}^2 - 456\,\cos(v)\,\sin(v)\,v^8\\
&\quad a_0{}^2\,b_0\,b_1{}^2 - 96\,v^7\,a_0{}^2\,a_1\,b_1{}^2 - 48\,v^3\,a_0\,a_1{}^2\,b_1\\
&- 96\,v^7\,a_0\,b_0{}^2\,b_1 - 384\,\cos(v)\,v^3\,a_0\,a_1\,b_1 - 576\,\cos(v)\,v^5\\
&\quad a_0\,b_0\,b_1 - 384\,\sin(v)b_0 - 192\,v^3\,b_0{}^2 + 12\,\sin(v)\,v^8\,a_0\,a_1
\end{aligned}
$$

$$
\begin{aligned}
& b_0{}^2\, b_1 + 384\, \cos(v)\, \sin(v)\, v^6\, a_0\, b_0\, b_1 + 48\, \cos(v) \\
& \sin(v)\, v^{10}\, a_0{}^2\, b_0\, b_1{}^2 + 36\, \sin(v)\, v^4\, a_0\, a_1{}^2\, b_1 \\
& + 192\, (\cos(v))^2\, v^7\, a_0{}^2\, a_1\, b_1{}^2 + 384\, v^7\, a_0{}^2\, b_0\, b_1{}^2 \\
& + 48\, \cos(v)\, \sin(v)\, v^4\, a_0\, a_1{}^2\, b_1 + 72\, \sin(v)\, v^6\, a_0\, a_1\, b_0\, b_1 \\
& + 480\, \cos(v)\, v^9\, a_0{}^3\, b_1{}^3 + 240\, \sin(v)\, v^{10}\, a_0{}^3\, b_1{}^3 \\
& + 288\, (\cos(v))^2\, v^5\, a_0\, a_1\, b_0\, b_1 - 384\, \cos(v)\, v^7\, a_0{}^2 \\
& b_1{}^2 - 192\, \cos(v)\, \sin(v)\, v^4\, a_0\, b_0\, b_1 - 96\, \cos(v) v\, a_1\, b_0{}^2 \\
& + 192\, \cos(v)\, \sin(v) a_1\, b_0 + 4\, \sin(v)\, v^{10}\, a_0\, b_0{}^3\, b_1 \\
& - 192\, \cos(v)\, v^3\, b_0{}^2 + 48\, \cos(v)\, \sin(v)\, v^4\, a_1\, b_0{}^2 \\
& + 96\, (\cos(v))^2\, v^3\, a_0\, a_1{}^2\, b_1 + 4\, \sin(v)\, v^4\, a_0\, a_1{}^3\, b_1 \\
& + 48\, (\cos(v))^2 v\, a_1\, b_0{}^2 - 192\, \cos(v)\, \sin(v)\, v^2\, b_0{}^2 \\
& + 12\, \sin(v)\, v^6\, a_0\, a_1{}^2\, b_0\, b_1 - 288\, \cos(v)\, \sin(v)\, v^6\, a_0 \\
& b_0{}^2\, b_1 - 48\, \cos(v)\, v^3\, a_1\, b_0{}^2 + 36\, \sin(v)\, v^2\, a_1{}^2\, b_0 \\
& + 12\, \sin(v)\, v^{10}\, a_0{}^2\, a_1\, b_0\, b_1{}^2 - 120\, v^9\, a_0{}^3\, a_1\, b_1{}^3 \\
& + 384\, \cos(v)\, v^5\, a_0\, b_0{}^2\, b_1 + \sin(v)\, v^{16}\, a_0{}^4\, b_1{}^4 \\
& - 768\, (\cos(v))^2 v\, a_0\, b_1 - 48\, \cos(v)\, v^{11}\, a_0{}^3\, b_1{}^3 \\
& - 2304\, \sin(v)\, v^2\, a_0\, b_1 + 48\, \cos(v)\, \sin(v)\, v^8\, a_0{}^2\, a_1\, b_1{}^2 \\
& + 1536\, \cos(v) v\, a_0\, b_1 - 576\, \cos(v)\, v^3\, a_0\, a_1\, b_0\, b_1 + 24\, v^9\, a_0{}^2 \\
& b_0{}^2\, b_1{}^2 + 240\, v^5\, a_0{}^2\, a_1{}^2\, b_1{}^2 + 4\, \sin(v)\, v^2\, a_1{}^3\, b_0 \\
& + \sin(v)\, a_1{}^4 - 192\, v\, b_0\, b_1 + 12\, \sin(v)\, a_1{}^3 + 768\, (\cos(v))^2 \\
& v^7\, a_0{}^2\, b_1{}^2 + 48\, v^3\, b_0{}^2\, b_1 - 960\, v^3\, a_0\, b_1{}^2 + 480\, v^7\, a_0{}^2\, b_1{}^3 \\
& + 3840\, v^5\, a_0{}^2\, b_1{}^2 + 96\, \cos(v)\, \sin(v)\, v^6\, a_0\, a_1\, b_0\, b_1 \\
& - 48\, \cos(v)\, \sin(v)\, v^4\, a_0\, a_1\, b_0\, b_1 + 384\, (\cos(v))^2\, v\, a_1 \\
& b_0 + 24\, \cos(v)\, \sin(v)\, a_1{}^2\, b_0 - 120\, \cos(v)\, v^9\, a_0{}^2\, b_0\, b_1{}^2 \\
& + 16\, \cos(v)\, \sin(v)\, v^6\, b_0{}^3 + 192\, v^3\, a_0\, a_1\, b_0\, b_1 \\
& - 384\, v^7\, a_0{}^2\, b_1{}^2 - 24\, \cos(v) v a_1{}^2\, b_0 - 24\, v\, a_0\, a_1{}^3\, b_1 \\
& - 12\, v\, a_1{}^2\, b_0\, b_1 + 1920\, v^5\, a_0{}^2\, a_1\, b_1{}^2
\end{aligned}
$$

Solving the above system of Eqs. (14)–(17), we can obtain the coefficients of the first block level of the proposed hybrid type four-step method:

$$
\begin{aligned}
a_0 &= \frac{F_5}{F_6}, \quad a_1 = \frac{F_7}{F_8} \\
b_0 &= \frac{F_9}{F_{10}}, \quad b_1 = \frac{F_{11}}{F_{10}}
\end{aligned} \tag{18}
$$

where

$$
\begin{aligned}
F_5 = {} & v^3 \sin(4v) - 14 v^3 \sin(2v) - 24 v^2 \cos(2v) \\
& + 5 v^2 \cos(4v) - 6 v \sin(4v) + 12 v \sin(2v) - 5 v^2 - 3 \cos(4v) + 3 \\
F_6 = {} & 2 \sin(4v) v^5 - 28 \sin(2v) v^5 - 72 v^4 \cos(2v) \\
& + 20 v^4 \cos(4v) - 56 v^3 \sin(4v) - 16 v^3 \sin(2v) \\
& + 4 \sin(5v) v^3 + 12 \sin(3v) v^3 + 8 \sin(v) v^3 - 44 v^4 \\
& + 12 v^2 \cos(5v) - 96 v^2 \cos(2v) + 60 v^2 \cos(3v) \\
& - 60 v^2 \cos(4v) + 120 \cos(v) v^2 + 24 v \sin(4v) + 48 v \sin(2v) \\
& - 12 v \sin(5v) - 36 v \sin(3v) - 24 \sin(v) v - 36 v^2 \\
F_7 = {} & 120 + 66 \sin(v) v^5 - 6 v^4 \cos(9v) - 151 v^2 \cos(7v) + 264 \cos(5v) v^4 \\
& + 328 \cos(3v) v^4 - 160 v^4 \cos(6v) - 26 v^4 \cos(8v) + 480 v \sin(5v) \\
& + 792 v \sin(3v) + 426 \sin(v) v + 96 \cos(5v) - v^5 \sin(9v) - 4 v^5 \sin(8v) \\
& + 13 v^2 \cos(9v) + 74 v^3 \sin(8v) + 124 \cos(v) v^4 + 112 v^2 \cos(8v) \\
& + 356 v^3 \sin(6v) + 123 v \sin(7v) - 155 v^3 \sin(7v) + 496 v^2 + 60 \sin(5v) v^5 \\
& + 116 \sin(3v) v^5 - 84 v \sin(8v) + 496 v^2 \cos(6v) - 32 \sin(6v) v^5 - 334 v^4 \\
& + 58 v^4 \cos(7v) - 168 \cos(v) + 9 v^5 \sin(7v) + 9 v \sin(9v) + 15 v^3 \sin(9v) \\
& + 96 \cos(2v) - 96 \cos(4v) - 360 v \sin(6v) - 88 \sin(4v) v^5 - 96 \sin(2v) v^5 \\
& - 608 v^4 \cos(2v) - 408 v^4 \cos(4v) + 928 v^2 \cos(4v) - 600 v \sin(4v) \\
& - 456 v \sin(2v) + 684 v^3 \sin(4v) + 596 v^3 \sin(2v) + 1040 v^2 \cos(2v) \\
& + 12 \cos(9v) - 96 \cos(6v) - 24 \cos(8v) + 60 \cos(7v) - 736 v^2 \cos(5v) \\
& - 1176 v^2 \cos(3v) - 760 \sin(5v) v^3 \\
& - 1200 \sin(3v) v^3 - 610 \sin(v) v^3 - 1022 \cos(v) v^2 \\
F_8 = {} & 60 - 58 v^2 \cos(7v) + 30 \cos(5v) v^4 + 122 \cos(3v) v^4 + 258 v \sin(5v) \\
& + 378 v \sin(3v) + 186 \sin(v) v + 72 \cos(5v) + 24 \cos(3v) - 4 v^3 \sin(8v) \\
& + 230 \cos(v) v^4 - 16 v^2 \cos(8v) + 72 v^3 \sin(6v) + 66 v \sin(7v) \\
& - 18 v^3 \sin(7v) + 368 v^2 + 80 v^2 \cos(6v) + 2 v^4 \cos(7v) - 120 \cos(v) \\
& + 48 \cos(2v) - 48 \cos(4v) - 96 v \sin(6v) + 416 v^2 \cos(4v) - 384 v \sin(4v) \\
& - 480 v \sin(2v) + 328 v^3 \sin(4v) + 424 v^3 \sin(2v) + 688 v^2 \cos(2v) \\
& - 48 \cos(6v) - 12 \cos(8v) + 24 \cos(7v) - 270 v^2 \cos(5v) - 538 v^2 \cos(3v) \\
& - 106 \sin(5v) v^3 - 210 \sin(3v) v^3 - 122 \sin(v) v^3 - 670 \cos(v) v^2 \\
F_9 = {} & 30 + 18 \sin(v) v^5 - 76 v^2 \cos(7v) + 55 \cos(5v) v^4 + 21 \cos(3v) v^4 \\
& - 40 v^4 \cos(6v) + 6 v^4 \cos(8v) + 144 v \sin(5v) + 144 v \sin(3v) \\
& + 48 \sin(v) v + 36 \cos(5v) + 12 \cos(3v) + v^5 \sin(8v) - 9 v^3 \sin(8v) \\
& - 93 \cos(v) v^4 + 5 v^2 \cos(8v) + 62 v^3 \sin(6v) + 48 v \sin(7v) \\
& - 55 v^3 \sin(7v) + 119 v^2 + 14 \sin(5v) v^5 + 30 \sin(3v) v^5 - 15 v \sin(8v) \\
& + 92 v^2 \cos(6v) - 10 \sin(6v) v^5 - 46 v^4 + 17 v^4 \cos(7v) - 60 \cos(v) \\
& + 2 v^5 \sin(7v) + 24 \cos(2v) - 24 \cos(4v) - 78 v \sin(6v) - 50 \sin(4v) v^5 \\
& - 66 \sin(2v) v^5 - 152 v^4 \cos(2v) - 152 v^4 \cos(4v) + 260 v^2 \cos(4v) \\
& - 162 v \sin(4v) - 150 v \sin(2v) + 194 v^3 \sin(4v) + 166 v^3 \sin(2v)
\end{aligned}
$$

$$+ 292\, v^2 \cos(2\, v) - 24 \cos(6\, v) - 6 \cos(8\, v) + 12 \cos(7\, v) - 228\, v^2 \cos(5\, v)$$
$$- 268\, v^2 \cos(3\, v) - 167 \sin(5\, v)\, v^3 - 171 \sin(3\, v)\, v^3 - 59 \sin(v)\, v^3$$
$$- 196 \cos(v)\, v^2$$

$$\begin{aligned} F_{10} = {} & 34\, v^6 + 86 \sin(v)\, v^5 - 6\, v^2 \cos(7\, v) + 48 \cos(5\, v)\, v^4 \\ & + 160 \cos(3\, v)\, v^4 - 29\, v^4 \cos(6\, v) + 184 \cos(v)\, v^4 + 33\, v^3 \sin(6\, v) \\ & + \cos(6\, v)\, v^6 - 24\, v^2 + 38 \sin(5\, v)\, v^5 + 126 \sin(3\, v)\, v^5 + 12\, v^2 \cos(6\, v) \\ & - 9 \sin(6\, v)\, v^5 - 86\, v^4 - 8\, v^4 \cos(7\, v) + 47 \cos(2\, v)\, v^6 + 14 \cos(4\, v)\, v^6 \\ & - 2\, v^5 \sin(7\, v) - 44 \sin(4\, v)\, v^5 - 61 \sin(2\, v)\, v^5 - 163\, v^4 \cos(2\, v) \\ & - 106\, v^4 \cos(4\, v) + 24\, v^2 \cos(4\, v) + 96\, v^3 \sin(4\, v) + 93\, v^3 \sin(2\, v) \\ & - 12\, v^2 \cos(2\, v) - 18\, v^2 \cos(5\, v) - 6\, v^2 \cos(3\, v) - 48 \sin(5\, v)\, v^3 \\ & - 144 \sin(3\, v)\, v^3 - 96 \sin(v)\, v^3 + 30 \cos(v)\, v^2 \end{aligned}$$

$$\begin{aligned} F_{11} = {} & 60 - 64\, v^2 \cos(7\, v) + 8\, v^4 \cos(6\, v) - 11\, v^4 \cos(8\, v) + 240\, v \sin(5\, v) \\ & + 336\, v \sin(3\, v) + 168 \sin(v)\, v + 48 \cos(5\, v) - v^5 \sin(8\, v) + 38\, v^3 \sin(8\, v) \\ & + 58\, v^2 \cos(8\, v) + 12\, v \sin(9\, v) + 104\, v^3 \sin(6\, v) + 84\, v \sin(7\, v) \\ & - 14\, v^3 \sin(7\, v) + 286\, v^2 - 42\, v \sin(8\, v) + 232\, v^2 \cos(6\, v) + 10 \sin(6\, v)\, v^5 \\ & + 239\, v^4 - 84 \cos(v) + 48 \cos(2\, v) - 48 \cos(4\, v) - 180\, v \sin(6\, v) \\ & + 6 \cos(9\, v) + 50 \sin(4\, v)\, v^5 + 66 \sin(2\, v)\, v^5 + 376\, v^4 \cos(2\, v) \\ & + 156\, v^4 \cos(4\, v) + 424\, v^2 \cos(4\, v) - 300\, v \sin(4\, v) - 228\, v \sin(2\, v) \\ & + 84\, v^3 \sin(4\, v) + 8\, v^3 \sin(2\, v) + 536\, v^2 \cos(2\, v) - 2\, v^3 \sin(9\, v) - 8\, v^2 \cos(9\, v) \\ & - 48 \cos(6\, v) - 12 \cos(8\, v) + 30 \cos(7\, v) - 232\, v^2 \cos(5\, v) - 504\, v^2 \cos(3\, v) \\ & - 40 \sin(5\, v)\, v^3 - 56 \sin(3\, v)\, v^3 - 28 \sin(v)\, v^3 - 728 \cos(v)\, v^2 \end{aligned}$$

For some values of $|w|$, the above mentioned formulae (18) may be subject to heavy cancelations. In this case, the following Taylor series expansions should be used:

$$\begin{aligned} a_0 = {} & \frac{9}{40} - \frac{47\, v^2}{1260} + \frac{379\, v^4}{14400} - \frac{187757\, v^6}{15523200} + \frac{129853819\, v^8}{24216192000} \\ & - \frac{38782494403\, v^{10}}{15256200960000} + \frac{1221159267631\, v^{12}}{1037421665280000} - \frac{222927836141455927\, v^{14}}{409791932002252800000} \\ & + \frac{110572400165192963\, v^{16}}{437111394135736320000} - \frac{53632306458652091 36297\, v^{18}}{45743707396304805888000000} + \cdots \end{aligned}$$

$$\begin{aligned} a_1 = {} & -2 + \frac{47\, v^8}{30240} - \frac{23\, v^{10}}{302400} + \frac{37\, v^{12}}{7983360} \\ & - \frac{62339\, v^{14}}{65383718400} + \frac{40501\, v^{16}}{1569209241600} \\ & + \frac{487397\, v^{18}}{166728481920000} + \cdots \end{aligned}$$

$$\begin{aligned} b_0 = {} & \frac{7}{6} - \frac{47\, v^4}{2520} - \frac{5\, v^6}{18144} + \frac{30347\, v^8}{239500800} + \frac{27361\, v^{10}}{9081072000} \\ & + \frac{563249\, v^{12}}{1307674368000} - \frac{22890841\, v^{14}}{190546836480000} - \frac{5096916017\, v^{16}}{491259059343360000} \end{aligned}$$

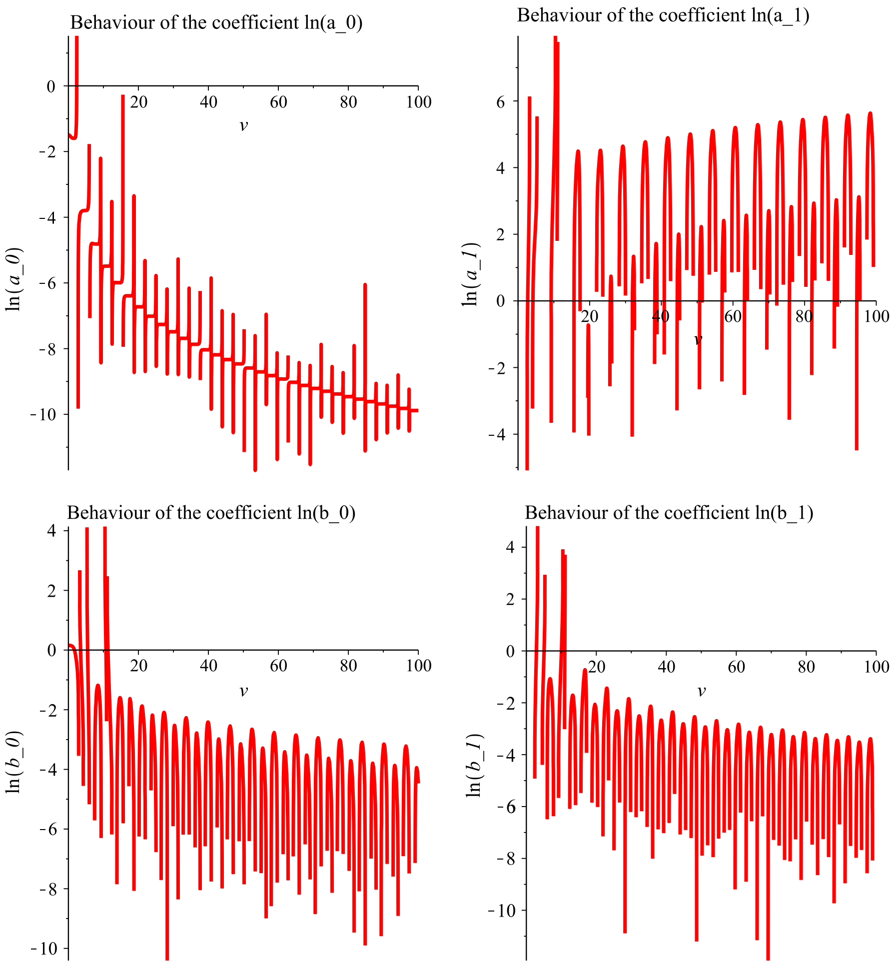


Fig. 4 Behavior of the coefficients of the method given by (18) for several values of $v = w\,h$

$$
\begin{aligned}
&- \frac{67530493843\, v^{18}}{15611121219133440000} + \cdots \\
b_1 = &- \frac{1}{3} + \frac{47\, v^4}{1260} - \frac{77\, v^6}{6480} + \frac{193063\, v^8}{119750400} - \frac{103703\, v^{10}}{1009008000} \\
&+ \frac{6997831\, v^{12}}{653837184000} - \frac{550389953\, v^{14}}{666913927680000} + \frac{67888619039\, v^{16}}{3193183885731840000} \\
&- \frac{942927663241\, v^{18}}{23416681828700160000} + \cdots
\end{aligned}
\qquad (19)
$$

Figure 4 shows the behavior of the coefficients a_0, a_1 b_0 and b_1.

3.2 Second level of the method

Let us now consider the second level of the hybrid type method (11) which consists the third level of the method:

$$p_{n+2} + a_2\, p_{n+1} + 2\, p_n + a_2\, p_{n-1} + p_{n-2} \\ = h^2 \left(b_4\, p''_{n+2} + b_3\, p''_{n+1} + b_2\, p''_n + b_3\, p''_{n-1} + b_4\, p''_{n-2} \right) \tag{20}$$

Applying the second level (20) to the scalar test Eq. (6), we obtain the difference Eq. (7) with $q = 2$ and $A_j(v),\ \ j = 0,\ 1,\ 2$ given by:

$$A_2(v) = 1 + v^2\, b_4, \quad A_1(v) = a_2 + v^2\, b_3 \\ A_0(v) = 2 + v^2\, b_2 \tag{21}$$

is produced.

Requiring now the above mentioned second level of the method (11) to have vanished the phase-lag and its first, second and third derivatives, the following system of equations is obtained [using the formula (10) and (21)]:

$$\text{Phase-Lag} = \frac{1}{2}\, \frac{F_{12}}{v^2 b_3 + 4\, v^2 b_4 + a_2 + 4} = 0 \tag{22}$$

$$\text{First Derivative of the Phase-Lag} = \frac{F_{13}}{\left(v^2\, b_3 + 4\, v^2\, b_4 + a_2 + 4\right)^2} = 0 \tag{23}$$

$$\text{Second Derivative of the Phase-Lag} = -\frac{F_{14}}{\left(v^2\, b_3 + 4\, v^2\, b_4 + a_2 + 4\right)^3} = 0 \tag{24}$$

$$\text{Third Derivative of the Phase-Lag} = -\frac{F_{15}}{\left(v^2\, b_3 + 4\, v^2\, b_4 + a_2 + 4\right)^4} = 0 \tag{25}$$

where

$$\begin{aligned} F_{12} &= 4\,(\cos(v))^2\, v^2\, b_4 + 2\, \cos(v)\, v^2\, b_3 + v^2\, b_2 \\ &\quad - 2\, v^2\, b_4 + 4\,(\cos(v))^2 + 2\, \cos(v)\, a_2 \\ F_{13} &= -4\, \cos(v)\, \sin(v)\, v^4\, b_3\, b_4 - 16\, \cos(v)\, \sin(v)\, v^4\, {b_4}^2 \\ &\quad - \sin(v)\, v^4\, {b_3}^2 - 4\, \sin(v)\, v^4\, b_3\, b_4 - 4\, \cos(v)\, \sin(v)\, v^2\, a_2\, b_4 \\ &\quad + 4\,(\cos(v))^2\, v\, a_2\, b_4 - 4\, \cos(v)\, \sin(v)\, v^2\, b_3 - 32\, \cos(v)\, v^2 \\ &\quad\ b_4\, \sin(v) - 2\, \sin(v)\, v^2\, a_2\, b_3 - 4\, \sin(v)\, v^2\, a_2\, b_4 \\ &\quad - 4\,(\cos(v))^2\, v\, b_3 - 8\, \cos(v)\, v\, a_2\, b_4 - 4\, \sin(v)\, v^2\, b_3 \\ &\quad - 4\, \cos(v)\, \sin(v)\, a_2 + 8\, \cos(v)\, v\, b_3 - \sin(v)\, {a_2}^2 + v\, a_2\, b_2 \\ &\quad - 2\, v\, a_2\, b_4 - 16\, \cos(v)\, \sin(v) - 4\, \sin(v)\, a_2 + 4\, v\, b_2 - 8\, v\, b_4 \\ F_{14} &= -64 - 4\, v^6\, {b_3}^2\, b_4 - 32\, v^4\, a_2\, {b_4}^2 - 32\, v^6\, b_3\, {b_4}^2 \\ &\quad + 12\,(\cos(v))^2\, v^2\, a_2\, b_3\, b_4 - 64\, \cos(v)\, \sin(v)\, v^3\, b_3\, b_4 \end{aligned}$$

$$
\begin{aligned}
& - 16\,\sin(v)\,v^3\,a_2\,b_3\,b_4 + 16\,\cos(v)\,\sin(v)\,v\,a_2{}^2\,b_4 \\
& - 24\,\cos(v)\,v^2\,a_2\,b_3\,b_4 + 64\,\cos(v)\,\sin(v)\,v^3\,a_2\,b_4{}^2 \\
& - 4\,a_2{}^2 + 8\,(\cos(v))^2\,a_2{}^2 + \cos(v)\,a_2{}^3 \\
& - a_2{}^2\,b_2 - 192\,v^2\,b_4 + 16\,\cos(v)a_2 - 32\,\cos(v)b_3 - 64\,v^6\,b_4{}^3 \\
& - 4\,v^4\,b_3{}^2 - 32\,v^2\,b_3 - 192\,v^4\,b_4{}^2 - 96\,v^2\,b_4{}^2 + 2\,a_2{}^2\,b_4 - 8\,a_2\,b_2 \\
& + 16\,a_2\,b_4 + 16\,(\cos(v))^2\,v^4\,a_2\,b_3\,b_4 - 8\,v^4\,a_2\,b_3\,b_4 \\
& + 32\,\cos(v)\,v^4\,b_3\,b_4 + 32\,\cos(v)\,v^2\,a_2\,b_4 + 3\,v^2\,a_2\,b_2\,b_3 \\
& + 12\,v^2\,a_2\,b_2\,b_4 - 6\,v^2\,a_2\,b_3\,b_4 + 8\,(\cos(v))^2\,v^6\,b_3{}^2\,b_4 \\
& + 64\,(\cos(v))^2\,v^6\,b_3\,b_4{}^2 + 8\,\cos(v)\,v^6\,b_3{}^2\,b_4 \\
& + 16\,\cos(v)\,v^6\,b_3\,b_4{}^2 + 64\,(\cos(v))^2\,v^4\,a_2\,b_4{}^2 \\
& + 128\,(\cos(v))^2\,v^4\,b_3\,b_4 + 3\,\cos(v)\,v^4\,a_2\,b_3{}^2 \\
& + 16\,\cos(v)\,v^4\,a_2\,b_4{}^2 + 8\,(\cos(v))^2\,v^2\,a_2{}^2\,b_4 \\
& + 48\,(\cos(v))^2\,v^2\,a_2\,b_4{}^2 - 16\,\cos(v)\,\sin(v)\,v^3\,b_3{}^2 \\
& - 64\,\sin(v)\,v^3\,a_2\,b_4{}^2 + 16\,(\cos(v))^2\,v^2\,a_2\,b_3 \\
& + 128\,(\cos(v))^2\,v^2\,a_2\,b_4 - 48\,(\cos(v))^2\,v^2\,b_3\,b_4 \\
& + 3\,\cos(v)\,v^2\,a_2{}^2\,b_3 + 8\,\cos(v)\,v^2\,a_2{}^2\,b_4 \\
& - 96\,\cos(v)\,v^2\,a_2\,b_4{}^2 + 64\,\sin(v)\,v^3\,b_3\,b_4 + 16\,\cos(v)\,v^2\,a_2\,b_3 \\
& + 96\,\cos(v)\,v^2\,b_3\,b_4 - 16\,\sin(v)\,v\,a_2{}^2\,b_4 - 64\,\cos(v)\,\sin(v)v\,b_3 \\
& + 16\,\sin(v)v\,a_2\,b_3 - 64\,\sin(v)v\,a_2\,b_4 + 16\,\cos(v)\,\sin(v)\,v^3\,a_2\,b_3\,b_4 \\
& + 16\,\cos(v)\,v^4\,a_2\,b_3\,b_4 + 64\,\cos(v)\,\sin(v)v\,a_2\,b_4 + 8\,\cos(v)\,a_2{}^2 \\
& - 16\,b_2 + 128\,(\cos(v))^2 + 64\,(\cos(v))^2\,a_2 + 16\,(\cos(v))^2\,b_3 \\
& - 16\,\cos(v)\,\sin(v)v\,a_2\,b_3 - 32\,a_2 + 32\,b_4 + 16\,\sin(v)\,v^3\,b_3{}^2 \\
& + 384\,(\cos(v))^2\,v^4\,b_4{}^2 + 8\,\cos(v)\,a_2{}^2\,b_4 + \cos(v)\,v^6\,b_3{}^3 \\
& - 4\,(\cos(v))^2\,a_2{}^2\,b_4 + 64\,(\cos(v))^2\,v^2\,b_3 \\
& - 16\,(\cos(v))^2\,a_2\,b_4 - 12\,(\cos(v))^2\,v^2\,b_3{}^2 \\
& + 384\,(\cos(v))^2\,v^2\,b_4 + 8\,\cos(v)\,v^4\,b_3{}^2 + 24\,\cos(v)\,v^2\,b_3{}^2 \\
& + 48\,v^2\,b_2\,b_4 + 32\,\cos(v)a_2\,b_4 - 24\,v^2\,b_3\,b_4 - 24\,v^2\,a_2\,b_4{}^2 \\
& + 4\,(\cos(v))^2\,a_2\,b_3 + 16\,\cos(v)\,v^2\,b_3 + 64\,\sin(v)v\,b_3 \\
& + 8\,(\cos(v))^2\,v^4\,b_3{}^2 - 8\,\cos(v)a_2\,b_3 - 64\,v^4\,b_3\,b_4 \\
& - 8\,v^2\,a_2\,b_3 - 64\,v^2\,a_2\,b_4 - 4\,v^2\,a_2{}^2\,b_4 \\
& + 12\,v^2\,b_2\,b_3 + 128\,(\cos(v))^2\,v^6\,b_4{}^3
\end{aligned}
$$

$$
\begin{aligned}
F_{15} = & -192\,\cos(v)\,\sin(v)\,a_2{}^2 - 12\,\sin(v)\,v^8\,b_3{}^3\,b_4 + 384\,v^3\,a_2\,b_4{}^3 \\
& - 48\,v^3\,b_2\,b_3{}^2 - 768\,v^3\,b_2\,b_4{}^2 - 96\,v\,a_2{}^2\,b_4{}^2 + 192\,vb_2\,b_3 + 768\,vb_2\,b_4 \\
& - 48\,(\cos(v))^2\,v^5\,b_3{}^3 - \sin(v)\,v^8\,b_3{}^4 - 12\,\sin(v)\,v^6\,b_3{}^3 \\
& - 4\,\sin(v)\,v^6\,a_2\,b_3{}^3 - 1152\,\cos(v)\,\sin(v)\,v^4\,a_2\,b_4{}^3 \\
& + 48\,\sin(v)\,v^2\,a_2{}^2\,b_3\,b_4 + 24\,\cos(v)\,v^5\,b_3{}^3 \\
& + 48\,(\cos(v))^2\,v^3\,b_3{}^3 - 72\,\sin(v)\,v^4\,b_3{}^3 \\
& - 1536\,\cos(v)\,v^3\,b_3\,b_4{}^2 - 64\,\sin(v)\,v^6\,a_2\,b_4{}^3 \\
& - 192\,\cos(v)\,v^3\,a_2{}^2\,b_4{}^2 - 192\,\cos(v)\,\sin(v)\,v^4\,b_3{}^2
\end{aligned}
$$

$$
\begin{aligned}
&- 384\, v b_3\, b_4 - 384\, \cos(v)\, \sin(v)\, v^2\, a_2{}^2\, b_4 - 1536\, v\, b_4{}^2 \\
&+ 768\, v^3\, b_3\, b_4{}^2 + 192\, v^5\, b_3{}^2\, b_4 - 384\, v^5\, a_2\, b_4{}^3 + 384\, v^5\, b_3\, b_4{}^2 \\
&- 576\, \sin(v)\, v^4\, b_3{}^2\, b_4 - 192\, v^5\, a_2\, b_3\, b_4{}^2 - 24\, v^5\, a_2\, b_3{}^2\, b_4 \\
&+ 96\, v^3\, b_3{}^2\, b_4 + 384\, v a_2\, b_2\, b_4 + 48\, \cos(v)\, v^3\, a_2\, b_3{}^2 \\
&+ 1536\, (\cos(v))^2\, v^3\, a_2\, b_4{}^2 + 72\, \sin(v)\, v^4\, a_2\, b_3{}^2\, b_4 \\
&- 96\, \sin(v)\, v^6\, a_2\, b_3\, b_4{}^2 + 768\, (\cos(v))^2 v a_2\, b_4{}^2 \\
&- 768\, (\cos(v))^2\, v^3\, a_2\, b_4{}^3 - 768\, \cos(v)\, \sin(v)\, v^6\, a_2\, b_4{}^3 \\
&- 48\, (\cos(v))^2\, v^3\, a_2\, b_3{}^2\, b_4 + 192\, \cos(v)\, \sin(v)\, a_2{}^2\, b_4 \\
&- 48\, \cos(v)\, \sin(v)\, v^6\, a_2\, b_3{}^2\, b_4 + 1536\, \cos(v)\, v^3\, a_2\, b_4{}^3 \\
&- 12\, v^3\, a_2\, b_2\, b_3{}^2 - 384\, \cos(v)\, \sin(v)\, v^6\, a_2\, b_3\, b_4{}^2 \\
&- 6\, \sin(v)\, v^4\, a_2{}^2\, b_3{}^2 - 192\, \sin(v)\, v^4\, a_2\, b_3\, b_4 - 192\, v\, a_2{}^2\, b_4 \\
&+ 1536\, \cos(v) v b_3\, b_4 + 768\, \sin(v)\, v^2\, a_2\, b_4{}^2 + 12\, v\, a_2{}^2\, b_2\, b_3 \\
&+ 48\, v^3\, a_2\, b_3{}^2 - 192\, v^3\, a_2{}^2\, b_4{}^2 + 384\, \cos(v)\, \sin(v) a_2\, b_4 \\
&+ 192\, v a_2\, b_3 - 768\, \sin(v)\, v^2\, b_3\, b_4 - 384\, v^3\, b_2\, b_3\, b_4 \\
&+ 48\, \cos(v)\, \sin(v)\, v^2\, a_2\, b_3{}^2 - 48\, \cos(v)\, \sin(v)\, v^4\, a_2\, b_3{}^2 \\
&+ 24\, v^5\, b_3{}^3 - 384\, \cos(v)\, \sin(v)\, v^6\, b_3{}^2\, b_4 - 96\, v^3\, a_2\, b_2\, b_3\, b_4 \\
&- 384\, \cos(v)\, \sin(v)\, v^2\, a_2\, b_3 + 192\, \cos(v)\, v^3\, b_3{}^2 + 48\, v\, a_2{}^2\, b_2\, b_4 \\
&- 576\, \cos(v)\, \sin(v)\, v^4\, a_2\, b_3\, b_4{}^2 \\
&- 384\, (\cos(v))^2\, v^3\, a_2\, b_3\, b_4{}^2 \\
&- 2304\, \cos(v)\, \sin(v)\, v^4\, b_3\, b_4 - 2304\, \cos(v)\, \sin(v)\, v^2\, a_2\, b_4 \\
&- 384\, \sin(v) a_2\, b_4 + 192\, \sin(v) a_2\, b_3 + 384\, (\cos(v))^2\, v^3\, b_3{}^2\, b_4 \\
&+ 1152\, \cos(v)\, \sin(v)\, v^4\, b_3\, b_4{}^2 - 24\, \cos(v)\, \sin(v)\, a_2{}^2\, b_3 \\
&+ 576\, \sin(v)\, v^4\, a_2\, b_3\, b_4{}^2 - 768\, \cos(v)\, v^3\, a_2\, b_4{}^2 \\
&- 384\, \cos(v)\, \sin(v) b_3 - 192\, \sin(v)\, a_2{}^2\, b_4 + 24\, \sin(v)\, a_2{}^2\, b_3 \\
&- 192\, \cos(v)\, v^5\, a_2\, b_3\, b_4{}^2 - 768\, \cos(v)\, v^3\, b_3{}^2\, b_4 - 192\, v^3\, a_2\, b_2\, b_4{}^2 \\
&+ 384\, v b_3 - 48\, \sin(v)\, a_2{}^2 - 48\, \sin(v)\, v^8\, b_3{}^2\, b_4{}^2 \\
&+ 48\, (\cos(v))^2\, v^5\, a_2\, b_3{}^2\, b_4 + 768\, (\cos(v))^2 v a_2\, b_4 \\
&- 768\, \cos(v)\, \sin(v)\, v^2\, b_3 - 96\, \sin(v)\, v^2\, a_2\, b_3 \\
&- 192\, \sin(v)\, v^2\, a_2\, b_4 - 384\, \cos(v) v a_2\, b_4 - 4096\, \cos(v)\, v^2\, b_4 \sin(v) \\
&- 6144\, \cos(v)\, \sin(v)\, v^4\, b_4{}^2 - 192\, \sin(v)\, v^4\, b_3\, b_4 \\
&+ 192\, (\cos(v))^2 v\, a_2{}^2\, b_4{}^2 + 768\, \cos(v)\, v^3\, b_3\, b_4 \\
&+ 384\, \cos(v) v\, b_3{}^2 - 24\, \sin(v)\, a_2{}^3\, b_4 - 192\, \sin(v)\, v^2\, b_3{}^2 \\
&- 768\, \cos(v)\, \sin(v)\, v^8\, b_3\, b_4{}^3 + 1536\, v^3\, b_4{}^3 - 768\, v^3\, a_2\, b_4{}^2 \\
&- 192\, v a_2\, b_3\, b_4 + 768\, v^3\, b_3\, b_4 - 36\, \sin(v)\, v^4\, a_2{}^2\, b_3\, b_4 \\
&- 192\, (\cos(v))^2 v\, b_3{}^2 - 96\, \cos(v)\, v^3\, b_3{}^3 \\
&- 384\, (\cos(v))^2\, v^3\, b_3{}^2 + 384\, (\cos(v))^2 v\, a_2{}^2\, b_4 \\
&- 12\, \sin(v)\, a_2{}^3 + 96\, v a_2\, b_2\, b_3 + 72\, \cos(v)\, \sin(v)\, v^4\, b_3{}^3 \\
&- 36\, \sin(v)\, v^4\, a_2\, b_3{}^2 + 48\, (\cos(v))^2 v\, a_2{}^2\, b_3\, b_4 \\
&- 48\, v^3\, a_2{}^2\, b_3\, b_4 - 192\, \cos(v)\, \sin(v) a_2\, b_3
\end{aligned}
$$

$$+ 768\,(\cos(v))^2\,v^3\,b_3\,b_4{}^2 - 768\,(\cos(v))^2 v b_3\,b_4$$
$$- 48\,(\cos(v))^2 v a_2\,b_3{}^2 + 384\,(\cos(v))^2\,v^5\,a_2\,b_3\,b_4{}^2$$
$$- 16\,\cos(v)\,\sin(v)\,v^8\,b_3{}^3\,b_4 - 48\,\sin(v)\,v^4\,a_2{}^2\,b_4{}^2$$
$$- 192\,\sin(v)\,v^4\,a_2\,b_4{}^2 + 24\,v^3\,a_2\,b_3{}^2\,b_4 - 24\,v\,a_2{}^3\,b_4 - 768\,v a_2\,b_4{}^2$$
$$- 768\,(\cos(v))^2 v b_3 - 768\,\cos(v)\,\sin(v) a_2 - 384\,v a_2\,b_4$$
$$- 48\,\sin(v)\,v^4\,b_3{}^2 - 64\,\sin(v)\,v^2\,b_3 + 384\,\cos(v) v b_3 + 24\,v\,a_2{}^2\,b_3$$
$$- 2304\,\cos(v)\,\sin(v)\,v^6\,b_3\,b_4{}^2 - 768\,\cos(v)\,\sin(v)\,v^2\,a_2\,b_4{}^2$$
$$- 48\,(\cos(v))^2 v\,a_2{}^2\,b_3 - 48\,\sin(v)\,v^2\,a_2\,b_3{}^2$$
$$- 1024\,\cos(v)\,\sin(v) - 192\,\cos(v)\,\sin(v)\,v^4\,a_2{}^2\,b_4{}^2$$
$$- 96\,\sin(v)\,v^2\,a_2{}^2\,b_4 - 36\,\sin(v)\,v^2\,a_2{}^2\,b_3$$
$$- 72\,\cos(v)\,\sin(v)\,v^4\,a_2\,b_3{}^2\,b_4 - 384\,\cos(v) v\,a_2{}^2\,b_4{}^2$$
$$- 24\,\cos(v) v\,a_2{}^3\,b_4 + 192\,\sin(v)\,v^2\,a_2{}^2\,b_4{}^2$$
$$- 384\,(\cos(v))^2\,v^5\,b_3{}^2\,b_4 - 48\,\cos(v)\,\sin(v)\,v^2\,a_2{}^2\,b_3$$
$$- 384\,(\cos(v))^2 v a_2\,b_3 - 48\,\cos(v)\,\sin(v)\,v^2\,a_2{}^2\,b_3\,b_4$$
$$- 768\,\cos(v)\,\sin(v)\,v^4\,a_2\,b_3\,b_4 + 96\,\cos(v) v a_2\,b_3{}^2$$
$$+ 24\,\cos(v) v\,a_2{}^2\,b_3 + 384\,\sin(v) b_3 - 192\,\cos(v) v\,a_2{}^2\,b_4$$
$$- \sin(v)\,a_2{}^4 - 16\,\cos(v)\,\sin(v)\,v^6\,b_3{}^3$$
$$- 36\,\sin(v)\,v^6\,a_2\,b_3{}^2\,b_4 - 12\,\sin(v)\,v^2\,a_2{}^3\,b_4$$
$$- 24\,\cos(v)\,v^5\,a_2\,b_3{}^2\,b_4 - 192\,\cos(v)\,\sin(v)\,v^8\,b_3{}^2\,b_4{}^2$$
$$+ 384\,(\cos(v))^2\,v^3\,a_2{}^2\,b_4{}^2 - 48\,\cos(v)\,\sin(v)\,v^4\,a_2{}^2$$
$$b_3\,b_4 + 1152\,\sin(v)\,v^4\,a_2\,b_4{}^3 - 96\,(\cos(v))^2\,v^3\,a_2\,b_3{}^2$$
$$+ 192\,\cos(v) v a_2\,b_3 + 96\,(\cos(v))^2\,v^3\,a_2{}^2\,b_3\,b_4$$
$$- 192\,\sin(v)\,v^6\,b_3\,b_4{}^2 - 96\,\sin(v)\,v^6\,b_3{}^2\,b_4$$
$$- 384\,\cos(v)\,v^5\,a_2\,b_4{}^3 - 768\,(\cos(v))^2\,v^5\,b_3\,b_4{}^2$$
$$- 16\,\cos(v)\,\sin(v)\,v^2\,a_2{}^3\,b_4 + 576\,\cos(v)\,\sin(v)\,v^4\,b_3{}^2\,b_4$$
$$- 2304\,\cos(v)\,\sin(v)\,v^4\,a_2\,b_4{}^2 - 64\,\sin(v)\,v^8\,b_3\,b_4{}^3$$
$$+ 768\,\cos(v)\,v^3\,a_2\,b_3\,b_4{}^2 + 48\,(\cos(v))^2 v\,a_2{}^3\,b_4$$
$$- 64\,\sin(v) a_2 - 4\,\sin(v)\,v^2\,a_2{}^3\,b_3 + 192\,v^3\,b_3{}^2$$
$$- 4096\,\cos(v)\,\sin(v)\,v^6\,b_4{}^3 - 1152\,\sin(v)\,v^4\,b_3\,b_4{}^2$$
$$- 1024\,\cos(v)\,\sin(v)\,v^8\,b_4{}^4 - 1536\,(\cos(v))^2\,v^3\,b_3\,b_4$$
$$+ 768\,(\cos(v))^2\,v^5\,a_2\,b_4{}^3 - 96\,\cos(v) v\,a_2{}^2\,b_3\,b_4$$
$$- 192\,\cos(v)\,\sin(v)\,v^2\,a_2{}^2\,b_4{}^2 + 192\,\cos(v)\,\sin(v)\,v^2\,b_3{}^2$$
$$+ 24\,\cos(v)\,\sin(v)\,a_2{}^3\,b_4 - 1536\,\cos(v) v a_2\,b_4{}^2$$
$$- 48\,\cos(v)\,v^3\,a_2{}^2\,b_3\,b_4 + 192\,\cos(v)\,v^5\,b_3{}^2\,b_4$$
$$- 24\,v\,a_2{}^2\,b_3\,b_4 + 384\,\cos(v)\,v^5\,b_3\,b_4{}^2 - 16\,\cos(v)\,\sin(v)\,a_2{}^3$$
$$+ 768\,\cos(v)\,\sin(v)\,v^2\,b_3\,b_4 + 96\,\cos(v)\,v^3\,a_2\,b_3{}^2\,b_4$$
$$+ 192\,v^3\,a_2\,b_3\,b_4{}^2$$

Solving the above system of Eqs. (22)–(25), we obtain the coefficients of the second level of the proposed hybrid four-step method:

$$a_2 = \frac{F_{16}}{F_{17}}, \quad b_2 = \frac{F_{22}}{F_{23}}$$
$$b_3 = \frac{F_{20}}{F_{21}}, \quad b_4 = \frac{F_{18}}{F_{19}} \tag{26}$$

where

$$\begin{aligned}
F_{16} = & -5612976\, v + 20736\, \sin(5\, v) - 242904\, \cos(10\, v) v^7 - 611472\, \sin(11\, v) v^6 \\
& - 311040\, \sin(9\, v) - 9984\, v^3 \cos(14\, v) - 1151136\, v^2 \sin(10\, v) \\
& + 8413632\, \cos(v) v - 17952\, v^3 \cos(11\, v) + 48624\, v^4 \sin(15\, v) + 4608372\, \cos(v) v^7 \\
& + 637836\, \cos(2\, v) v^5 - 67392\, v \cos(13\, v) - 1052172\, v^8 \sin(6\, v) + 103168\, \cos(v) v^9 \\
& + 22977048\, v^4 \sin(6\, v) - 90330\, v^6 \sin(12\, v) + 2075328\, v \cos(9\, v) + 15984\, v \cos(6\, v) \\
& + 120920\, \sin(3\, v) v^8 + 568032\, v^3 \cos(7\, v) - 648\, v^7 \cos(14\, v) - 35840\, v^9 \cos(7\, v) \\
& + 8\, v^9 \cos(14\, v) - 1096\, \sin(13\, v) v^8 - 442240\, v^9 - 15414372\, v^5 \cos(5\, v) \\
& + 18657792\, v^2 \sin(v) - 1688256\, v \cos(10\, v) + 938304\, v \cos(11\, v) \\
& + 4432320\, \cos(6\, v) v + 249280\, v^8 \sin(8\, v) + 37093680\, \sin(v) v^4 + 15554096\, \sin(v) v^6 \\
& + 24385536\, \sin(3\, v) v^2 - 6315990\, v^7 + 933120\, \sin(v) - 111976\, \sin(9\, v) v^8 \\
& + 12192000\, \cos(6\, v) v^3 - 105984\, v^2 \sin(15\, v) - 11840016\, \sin(5\, v) v^4 \\
& + 4877184\, \cos(4\, v) v^7 - 18820512\, v^3 \cos(3\, v) + 910\, v^6 \sin(14\, v) - 192\, \cos(12\, v) v^9 \\
& + 105408\, \cos(14\, v) v - 13873872\, \sin(7\, v) v^4 - 20924604\, v^5 \cos(3\, v) - 4253054\, v^6 \sin(8\, v) \\
& + 6891936\, v^3 \cos(9\, v) - 721596\, v^7 \cos(9\, v) - 194460\, v^7 \cos(5\, v) + 812808\, v^4 \sin(8\, v) \\
& - 4342344\, v^7 \cos(6\, v) + 4585896\, \cos(2\, v) v^7 + 1612800\, \sin(11\, v) v^2 - 2202624\, \sin(5\, v) v^2 \\
& + 6574598\, v^6 \sin(6\, v) - 3077568\, v \cos(3\, v) + 6895008\, v^2 \sin(8\, v) - 18\, v^7 \cos(6\, v) \\
& - 124\, v^8 \sin(14\, v) + 3088\, v^8 \sin(12\, v) - 274400\, \cos(12\, v) v^5 + 2382860\, \cos(10\, v) v^5 \\
& - 672\, v^9 \cos(11\, v) + 1878188\, v^5 \cos(9\, v) + 6400\, v^9 \cos(9\, v) - 1004624\, v^6 \sin(7\, v) \\
& + 2930160\, v^4 \sin(9\, v) - 11198880\, v^2 \sin(4\, v) - 659424\, v^3 \cos(13\, v) + 24\, v^8 \sin(15\, v) \\
& + 21168\, v^2 \sin(6\, v) + 72288\, v^2 \sin(14\, v) - 5544612\, v^7 \cos(3\, v) - 10038528\, \cos(2\, v) v^3 \\
& - 51840\, \sin(14\, v) - 134784\, \sin(12\, v) + 77296\, v^6 \sin(13\, v) + 10193456\, \sin(3\, v) v^6 \\
& + 11821536\, v^2 \sin(6\, v) - 1397952\, \cos(8\, v) v - 60731304\, v^4 \sin(2\, v) + 1727628\, v^7 \cos(7\, v) \\
& - 189856\, v^9 \cos(3\, v) + 32\, v^9 \cos(13\, v) + 1863\, v^5 \cos(6\, v) + 2392\, v^9 \cos(10\, v) \\
& - 36412\, v^8 \sin(10\, v) + 7202304\, \cos(4\, v) v + 447160\, \sin(7\, v) v^8 + 21504\, v^7 \cos(12\, v) \\
& - 3540744\, v^4 \sin(10\, v) - 29181024\, v^3 \cos(5\, v) + 37544640\, \cos(4\, v) v^3 - 756864\, \sin(4\, v) \\
& - 26068778\, v^6 \sin(2\, v) + 39845616\, \sin(3\, v) v^4 + 90720\, v^3 \cos(15\, v) + 2063216\, \sin(9\, v) v^6 \\
& - 1454156\, v^5 \cos(11\, v) - 29781528\, v^3 - 834728\, v^8 \sin(5\, v) + 1019712\, \cos(12\, v) v^3 \\
& + 2539600\, v^8 \sin(4\, v) + 444\, v^7 \cos(15\, v) + 279\, v^6 \sin(6\, v) - 27030971\, v^5 - 6948\, v^4 \sin(6\, v) \\
& - 2345472\, v^2 \sin(9\, v) + 31104\, \sin(10\, v) - 6972480\, v \cos(5\, v) - 8228808\, v^4 \sin(4\, v) \\
& - 207360\, \cos(12\, v) v - 9120848\, \sin(5\, v) v^6 - 30214944\, v^2 \sin(2\, v) - 4590460\, \cos(6\, v) v^5 \\
& + 20736\, \sin(15\, v) + 1347840\, \sin(3\, v) + 103680\, \sin(13\, v) + 15544\, v^8 \sin(11\, v) \\
& + 1531448\, \sin(v) v^8 + 7645316\, v^5 \cos(7\, v) + 1417320\, \cos(8\, v) v^7 - 70848\, v \cos(15\, v) \\
& - 2755404\, v^8 \sin(2\, v) - 3100\, v^5 \cos(14\, v) + 590976\, \sin(8\, v) + 570240\, \sin(6\, v)
\end{aligned}$$

$$- 560976 \sin(13 v)v^4 - 2143488 \cos(10 v)v^3 + 9528 v^4 \sin(14 v) - 6201676 v^5 \cos(8 v)$$
$$- 11450880 \sin(7 v)v^2 + 953150 v^6 \sin(10 v) + 331776 \sin(13 v)v^2 + 31932320 \cos(4 v)v^5$$
$$- 8767200 v^3 \cos(8 v) - 18560 \cos(8 v)v^9 + 637464 v^4 \sin(12 v) + 7989550 v^6 \sin(4 v)$$
$$+ 116768 v^9 \cos(5 v) + 95176 \cos(6 v)v^9 - 325440 \cos(4 v)v^9 - 725760 \sin(7 v)$$
$$+ 103680 \sin(11 v) + 137100 v^7 \cos(11 v) + 41128224 \cos(v)v^3 - 2849472 \cos(2 v)v$$
$$- 891936 v^2 \sin(12 v) - 12876 v^7 \cos(13 v) - 1503360 \sin(2 v) - 5184 \sin(6 v)$$
$$+ 688856 \cos(2 v)v^9 + 1602480 \sin(11 v)v^4 - 1238976 v \cos(7 v) - 3600 v^6 \sin(15 v)$$
$$- 16716 v^5 \cos(15 v) + 269420 v^5 \cos(13 v) + 31162652 \cos(v)v^5 - 15624 v^3 \cos(6 v)$$
$$F_{17} = -3138048 v - 98496 \sin(5 v) + 37176 \cos(10 v)v^7 - 5717 \sin(11 v)v^6$$
$$- 119232 \sin(9 v) - 31104 v^3 \cos(14 v) - 682560 v^2 \sin(10 v) + 5126976 \cos(v)v$$
$$- 449232 v^3 \cos(11 v) + 252 v^4 \sin(15 v) + 2440293 \cos(v)v^7 - 1098176 \cos(2 v)v^5$$
$$- 62208 v \cos(13 v) + 31772 v^8 \sin(6 v) + 38320 \cos(v)v^9 + 8797968 v^4 \sin(6 v)$$
$$+ 13088 v^6 \sin(12 v) + 1232064 v \cos(9 v) - 303692 \sin(3 v)v^8 + 2637264 v^3 \cos(7 v)$$
$$+ 168 v^7 \cos(14 v) - 10264 v^9 \cos(7 v) - 44 \sin(13 v)v^8 - 20256 v^9$$
$$- 4835472 v^5 \cos(5 v) + 11653776 v^2 \sin(v) - 857088 v \cos(10 v)$$
$$+ 300672 v \cos(11 v) + 2239488 \cos(6 v)v - 39744 v^8 \sin(8 v) + 22172940 \sin(v)v^4$$
$$+ 9656855 \sin(v)v^6 + 13429008 \sin(3 v)v^2 - 1384800 v^7 + 648000 \sin(v)$$
$$- 14568 \sin(9 v)v^8 + 5630592 \cos(6 v)v^3 - 10800 v^2 \sin(15 v) - 9252564 \sin(5 v)v^4$$
$$+ 1412496 \cos(4 v)v^7 - 13284816 v^3 \cos(3 v) - 828 v^6 \sin(14 v) + 16 \cos(12 v)v^9$$
$$+ 55296 \cos(14 v)v - 4389396 \sin(7 v)v^4 - 13663880 v^5 \cos(3 v) - 713088 v^6 \sin(8 v)$$
$$+ 2377584 v^3 \cos(9 v) - 49071 v^7 \cos(9 v) + 933933 v^7 \cos(5 v) + 857856 v^4 \sin(8 v)$$
$$- 604056 v^7 \cos(6 v) + 566712 \cos(2 v)v^7 + 673488 \sin(11 v)v^2 - 3204720 \sin(5 v)v^2$$
$$+ 2326292 v^6 \sin(6 v) - 2433024 v \cos(3 v) + 3188736 v^2 \sin(8 v)$$
$$+ 12 v^8 \sin(14 v) - 560 v^8 \sin(12 v) - 16880 \cos(12 v)v^5 + 469120 \cos(10 v)v^5$$
$$- 124 v^9 \cos(11 v) - 553356 v^5 \cos(9 v) + 1512 v^9 \cos(9 v) + 1027727 v^6 \sin(7 v)$$
$$+ 1902300 v^4 \sin(9 v) - 5313024 v^2 \sin(4 v) - 17520 v^3 \cos(13 v) + 59328 v^2 \sin(14 v)$$
$$- 3402213 v^7 \cos(3 v) - 4735104 \cos(2 v)v^3 - 20736 \sin(14 v) - 82944 \sin(12 v)$$
$$- 1165 v^6 \sin(13 v) + 3690471 \sin(3 v)v^6 + 6121152 v^2 \sin(6 v) - 953856 \cos(8 v)v$$
$$- 24916464 v^4 \sin(2 v) + 72723 v^7 \cos(7 v) - 68292 v^9 \cos(3 v) + 4 v^9 \cos(13 v)$$
$$- 448 v^9 \cos(10 v) + 7948 v^8 \sin(10 v) + 4126464 \cos(4 v)v + 76712 \sin(7 v)v^8$$
$$- 4752 v^7 \cos(12 v) - 1230000 v^4 \sin(10 v) - 14043888 v^3 \cos(5 v) + 17052288 \cos(4 v)v^3$$
$$- 414720 \sin(4 v) - 7782092 v^6 \sin(2 v) + 19329612 \sin(3 v)v^4 + 4176 v^3 \cos(15 v)$$
$$+ 3867 \sin(9 v)v^6 + 9568 v^5 \cos(11 v) - 13711104 v^3 - 119204 v^8 \sin(5 v)$$
$$+ 402816 \cos(12 v)v^3 + 228624 v^8 \sin(4 v) - 9 v^7 \cos(15 v) - 9764640 v^5$$
$$- 217008 v^2 \sin(9 v) - 20736 \sin(10 v) - 4053888 v \cos(5 v) - 4611264 v^4 \sin(4 v)$$
$$- 34560 \cos(12 v)v - 4946977 \sin(5 v)v^6 - 15365952 v^2 \sin(2 v) - 156416 \cos(6 v)v^5$$
$$+ 5184 \sin(15 v) + 855360 \sin(3 v) + 46656 \sin(13 v) + 1276 v^8 \sin(11 v) + 1087760 \sin(v)v^8$$
$$+ 4038700 v^5 \cos(7 v) - 22944 \cos(8 v)v^7 - 12096 v \cos(15 v) - 430052 v^8 \sin(2 v)$$
$$- 960 v^5 \cos(14 v) + 331776 \sin(8 v) + 393984 \sin(6 v) - 5316 \sin(13 v)v^4$$
$$- 864384 \cos(10 v)v^3 - 6576 v^4 \sin(14 v) - 2183136 v^5 \cos(8 v) - 5869296 \sin(7 v)v^2$$
$$+ 4516 v^6 \sin(10 v) - 9648 \sin(13 v)v^2 + 11178224 \cos(4 v)v^5 - 3744000 v^3 \cos(8 v)$$
$$+ 4128 \cos(8 v)v^9 + 178752 v^4 \sin(12 v) + 993696 v^6 \sin(4 v) + 38844 v^9 \cos(5 v)$$

$$
\begin{aligned}
&- 15040\,\cos(6\,v)v^9 + 16112\,\cos(4\,v)v^9 - 471744\,\sin(7\,v) + 88128\,\sin(11\,v)\\
&+ 4335\,v^7\cos(11\,v) + 22776432\,\cos(v)v^3 - 1437696\,\cos(2\,v)v - 354816\,v^2\sin(12\,v)\\
&+ 9\,v^7\cos(13\,v) - 933120\,\sin(2\,v) + 15488\,\cos(2\,v)v^9 - 123108\,\sin(11\,v)v^4\\
&- 98496\,v\cos(7\,v) + 99\,v^6\sin(15\,v) + 324\,v^5\cos(15\,v) - 2504\,v^5\cos(13\,v)\\
&+ 16579484\,\cos(v)v^5
\end{aligned}
$$

$$
\begin{aligned}
F_{18} = {} & 12096 - 168\,v^6\cos(12\,v) - 160132\,v^5\sin(4\,v) - 40032\,v\sin(6\,v) - 58326\,v^3\sin(9\,v)\\
&+ 30360\,v^7\sin(v) - 16\,v^8\cos(10\,v) + 1170\,v^2\cos(13\,v) + 7128\,\cos(3\,v) + 15768\,\cos(5\,v)\\
&- 215136\,v^4\cos(5\,v) + 64904\,v^5\sin(6\,v) + 20784\,v^6\cos(6\,v) + 90792\,v^6\cos(4\,v)\\
&- 14688\,\cos(4\,v) + 186448\,v^4\cos(6\,v) + 100\,v^8\cos(9\,v) - 5688\,v^2\cos(12\,v)\\
&+ 242961\,v^3\sin(5\,v) + 4320\,v\sin(10\,v) - 22464\,v\sin(8\,v) + 65616\,v^2\cos(8\,v)\\
&- 1060\,v^7\sin(9\,v) - 908\,v^8\cos(7\,v) - 76308\,v^2\cos(7\,v) + 112836\,\cos(v)v^6\\
&- 62385\,v^6\cos(3\,v) + 864\,\cos(12\,v) + 39528\,v\sin(7\,v) - 1944\,\cos(11\,v) + 139\,v^5\sin(11\,v)\\
&- 432\,v^4\cos(13\,v) - 25008\,v^6\cos(8\,v) + 4080\,v^6\cos(10\,v) - 5412\,v^3\sin(12\,v)\\
&+ 53040\,v^2 - 78000\,v^3\sin(2\,v) - 33780\,v^2\cos(9\,v) + 126942\,v^2\cos(3\,v)\\
&+ 3096\,v\sin(9\,v) - 12\,v^7\sin(12\,v) + 47046\,v^6\cos(7\,v) - 5610\,v^6\cos(9\,v)\\
&+ 1728\,\cos(8\,v) - 235956\,v^3\sin(4\,v) - 3888\,\cos(9\,v) - 37084\,v^7\sin(5\,v)\\
&- 99\,v^5\sin(13\,v) + 20075\,v^5\sin(5\,v) + 31288\,v^7\sin(3\,v) + 172299\,v^3\sin(3\,v)\\
&+ 280541\,v^5\sin(3\,v) + 3012\,v^4\cos(12\,v) + 131692\,v^5\sin(v) + 2688\,v^8\cos(4\,v)\\
&+ 9\,v^6\cos(13\,v) - 2000\,v^8\cos(6\,v) + 52368\,v^3\sin(8\,v) + 14968\,v^7\sin(6\,v)\\
&- 86004\,v\sin(3\,v) - 29952\,v^2\cos(2\,v) - 24864\,v^6\cos(2\,v) + 963\,v^3\sin(13\,v)\\
&+ 6912\,\cos(2\,v) - 19872\,\cos(v) - 185096\,v^4 - 9640\,v^8\cos(3\,v) + 29064\,v^3\sin(10\,v)\\
&+ 5658\,v^2\cos(11\,v) + 3312\,v\sin(12\,v) + 45872\,v^4\cos(9\,v) + 128\,v^4\cos(11\,v)\\
&- 4\,v^8\cos(11\,v) - 10368\,\cos(6\,v) + 5736\,\cos(v)v^8 + 20042\,v^5\sin(9\,v)\\
&- 108970\,v^5\sin(7\,v) - 756\,v\sin(13\,v) - 254688\,v^4\cos(2\,v) + 2016\,v^8\cos(2\,v)\\
&- 3040\,v^8 - 120744\,v^3\sin(6\,v) + 4716\,v^8\cos(5\,v) + 20352\,v^2\cos(6\,v)\\
&- 17700\,v^3\sin(v) + 9600\,v^2\cos(10\,v) - 73008\,v\sin(v) + 55350\,v^2\cos(5\,v)\\
&+ 137360\,v^4\cos(3\,v) - 79032\,\cos(v)v^2 - 65616\,v^6 + 3024\,\cos(7\,v)\\
&+ 98496\,v\sin(2\,v) + 34992\,v\sin(4\,v) + 3456\,\cos(10\,v) - 106640\,v^4\cos(7\,v)\\
&+ 44\,v^7\sin(11\,v) - 112968\,v^2\cos(4\,v) - 30064\,v^4\cos(10\,v) - 240272\,v^5\sin(2\,v)\\
&- 5292\,v\sin(11\,v) + 64764\,v^4\cos(4\,v) + 19016\,v^4\cos(8\,v) + 18900\,v\sin(5\,v)\\
&- 91989\,v^6\cos(5\,v) - 6618\,v^3\sin(7\,v) + 4292\,v^7\sin(4\,v) - 35056\,v^7\sin(2\,v)\\
&+ 93\,v^6\cos(11\,v) + 488\,v^7\sin(10\,v) - 5200\,v^7\sin(8\,v) + 352\,v^8\cos(8\,v)\\
&+ 10036\,v^7\sin(7\,v) + 972\,v^5\sin(12\,v) - 15464\,v^5\sin(10\,v) + 47888\,v^5\sin(8\,v)\\
&- 216\,\cos(13\,v) + 335456\,\cos(v)v^4 + 2217\,v^3\sin(11\,v)
\end{aligned}
$$

$$
\begin{aligned}
F_{19} = {} & 768\,v^6\cos(12\,v) + 12\,v^9\sin(12\,v) + 75072\,v^5\sin(4\,v) - 1728\,v^3\sin(9\,v)\\
&- 664624\,v^7\sin(v) - 1872\,v^8\cos(10\,v) - 24\,v^9\sin(11\,v) + 189072\,v^4\cos(5\,v)\\
&- 295872\,v^5\sin(6\,v) + 908\,v^{10}\cos(7\,v) + 5472\,v^6\cos(6\,v) - 559104\,v^6\cos(4\,v)\\
&- 5688\,v^9\sin(7\,v) - 58752\,v^4\cos(6\,v) + 1194\,v^8\cos(9\,v) - 25668\,v^9\sin(4\,v)\\
&+ 21600\,v^3\sin(5\,v) - 9\,v^8\cos(13\,v) - 4716\,v^{10}\cos(5\,v) - 100\,v^{10}\cos(9\,v)\\
&+ 13176\,v^9\sin(5\,v) + 400\,v^7\sin(9\,v) - 4998\,v^8\cos(7\,v) - 734832\,\cos(v)v^6\\
&+ 658608\,v^6\cos(3\,v) - 5736\,v^{10}\cos(v) - 352\,v^{10}\cos(8\,v) + 600\,v^9\sin(9\,v)\\
&- 1200\,v^5\sin(11\,v) + 1296\,v^4\cos(13\,v) + 110400\,v^6\cos(8\,v) - 20256\,v^6\cos(10\,v)
\end{aligned}
$$

$$
\begin{aligned}
& + 3456\, v^3 \sin(12\, v) + 69120\, v^3 \sin(2\, v) - 216\, v^7 \sin(12\, v) - 189240\, v^6 \cos(7\, v) \\
& + 16680\, v^6 \cos(9\, v) + 17280\, v^3 \sin(4\, v) + 335878\, v^7 \sin(5\, v) + 16\, v^{10} \cos(10\, v) \\
& - 2688\, v^{10} \cos(4\, v) + 576\, v^5 \sin(13\, v) + 305904\, v^5 \sin(5\, v) - 271010\, v^7 \sin(3\, v) \\
& - 56160\, v^3 \sin(3\, v) - 499488\, v^5 \sin(3\, v) - 6336\, v^4 \cos(12\, v) - 601824\, v^5 \sin(v) \\
& - 140076\, v^8 \cos(4\, v) - 2016\, v^{10} \cos(2\, v) + 4\, v^{10} \cos(11\, v) + 54\, v^7 \sin(13\, v) \\
& + 50928\, v^8 \cos(6\, v) + 2000\, v^{10} \cos(6\, v) - 13824\, v^3 \sin(8\, v) + 108\, v^8 \cos(12\, v) \\
& + 9640\, v^{10} \cos(3\, v) - 168224\, v^7 \sin(6\, v) + 3600\, v^9 \sin(8\, v) + 14784\, v^6 \cos(2\, v) \\
& - 864\, v^3 \sin(13\, v) + 180864\, v^4 + 317361\, v^8 \cos(3\, v) + 54672\, v^9 \sin(2\, v) \\
& + 6912\, v^3 \sin(10\, v) - 49536\, v^4 \cos(9\, v) + 3888\, v^4 \cos(11\, v) + 99\, v^8 \cos(11\, v) \\
& - 238020 \cos(v) v^8 - 53808\, v^5 \sin(9\, v) + 151536\, v^5 \sin(7\, v) + 39168\, v^4 \cos(2\, v) \\
& - 49056\, v^8 \cos(2\, v) + 137064\, v^8 - 34560\, v^3 \sin(6\, v) - 75627\, v^8 \cos(5\, v) \\
& - 51840\, v^3 \sin(v) + 202032\, v^4 \cos(3\, v) + 447936\, v^6 - 408\, v^9 \sin(10\, v) \\
& - 133008\, v^9 \sin(v) - 59904\, v^4 \cos(7\, v) - 418\, v^7 \sin(11\, v) + 19584\, v^4 \cos(10\, v) \\
& + 686976\, v^5 \sin(2\, v) + 33936\, v^9 \sin(3\, v) - 251712\, v^4 \cos(4\, v) + 77184\, v^4 \cos(8\, v) \\
& + 3040\, v^{10} - 5256\, v^9 \sin(6\, v) + 248280\, v^6 \cos(5\, v) + 29376\, v^3 \sin(7\, v) \\
& - 63416\, v^7 \sin(4\, v) + 557504\, v^7 \sin(2\, v) + 504\, v^6 \cos(11\, v) - 736\, v^7 \sin(10\, v) \\
& + 44320\, v^7 \sin(8\, v) + 2904\, v^8 \cos(8\, v) - 56864\, v^7 \sin(7\, v) - 4032\, v^5 \sin(12\, v) \\
& + 40128\, v^5 \sin(10\, v) - 31488\, v^5 \sin(8\, v) - 286848 \cos(v) v^4 - 6048\, v^3 \sin(11\, v)
\end{aligned}
$$

$$
\begin{aligned}
F_{20} = & -1005048\, v + 5184 \sin(15\, v) - 597024\, v \cos(3\, v) + 92\, v^8 \sin(14\, v) \\
& - 2192\, v^8 \sin(12\, v) + 36928 \cos(12\, v) v^5 - 187660 \cos(10\, v) v^5 + 672\, v^9 \cos(11\, v) \\
& + 1571256\, v^2 \sin(6\, v) - 1203256 \sin(v) v^8 - 182616\, v^3 \cos(11\, v) + 7776 \sin(10\, v) \\
& + 147744 \sin(8\, v) + 630744\, v^3 \cos(9\, v) + 1948278\, v^3 - 121048 \sin(3\, v) v^8 + 5184 \sin(5\, v) \\
& + 2130348\, v^8 \sin(2\, v) - 47428\, v^5 \cos(13\, v) + 235728\, v^2 \sin(9\, v) + 20244\, v^4 \sin(12\, v) \\
& - 28512\, v \cos(13\, v) - 1241568\, v \cos(5\, v) + 9096\, v^3 \cos(14\, v) + 189856\, v^9 \cos(3\, v) \\
& - 158112\, v \cos(7\, v) - 95176 \cos(6\, v) v^9 + 325440 \cos(4\, v) v^9 + 152928\, v \cos(11\, v) \\
& - 517608\, v^2 \sin(10\, v) + 2916\, v^2 \sin(6\, v) + 9432\, v^2 \sin(14\, v) + 8865055\, v^5 \\
& + 2664\, v^3 \cos(15\, v) + 359004\, v^7 \cos(9\, v) + 2657880 \cos(2\, v) v^3 + 306\, v^3 \cos(6\, v) \\
& - 1278340 \cos(6\, v) v^5 + 182304 \sin(11\, v) v^6 - 5563616 \sin(3\, v) v^6 - 332216 \sin(7\, v) v^8 \\
& + 83048 \sin(9\, v) v^8 - 688856 \cos(2\, v) v^9 + 3694038\, v^7 + 233280 \sin(v) - 175040\, v^8 \sin(8\, v) \\
& - 23712 \sin(11\, v) v^4 - 1015920 \sin(5\, v) v^2 + 1191662\, v^6 \sin(8\, v) + 3243972\, v^7 \cos(3\, v) \\
& + 18\, v^7 \cos(6\, v) + 2232600\, v^7 \cos(6\, v) - 2918030\, v^6 \sin(6\, v) - 657824 \sin(9\, v) v^6 \\
& + 442240\, v^9 + 746496 \cos(6\, v) v - 24\, v^8 \sin(15\, v) + 403488\, v \cos(9\, v) + 2016\, v^6 \sin(15\, v) \\
& - 7016096 \sin(v) v^6 - 238310\, v^6 \sin(10\, v) - 1296 \sin(6\, v) + 968 \sin(13\, v) v^8 \\
& + 1321488 \sin(3\, v) v^2 + 1052\, v^5 \cos(14\, v) - 6816\, v^4 \sin(15\, v) + 18560 \cos(8\, v) v^9 \\
& + 5707884\, v^5 \cos(5\, v) - 6400\, v^9 \cos(9\, v) + 280440\, v^2 \sin(8\, v) - 1674238\, v^6 \sin(4\, v) \\
& - 1222812\, v^4 \sin(6\, v) + 653608\, v^8 \sin(5\, v) - 12344\, v^8 \sin(11\, v) + 839648\, v^6 \sin(7\, v) \\
& - 2392\, v^9 \cos(10\, v) + 25628\, v^8 \sin(10\, v) + 192 \cos(12\, v) v^9 - 8\, v^9 \cos(14\, v) \\
& - 48\, v^7 \cos(15\, v) - 207\, v^6 \sin(6\, v) + 458\, v^6 \sin(14\, v) - 9973536 \sin(3\, v) v^4 \\
& - 187992\, v^2 \sin(12\, v) + 8364\, v^7 \cos(13\, v) - 1057392 \sin(7\, v) v^2 - 11995828 \cos(v) v^5 \\
& - 15552 \cos(12\, v) v - 470016 \cos(2\, v) v - 1974816 \sin(5\, v) v^4 + 3165344 \sin(5\, v) v^6 \\
& - 375840 \sin(2\, v) + 35840\, v^9 \cos(7\, v) - 6839328 \sin(v) v^4 - 603940\, v^5 \cos(9\, v) \\
& - 8568\, v^2 \sin(4\, v) + 165412\, v^5 \cos(11\, v) - 103168 \cos(v) v^9 - 9984\, v^7 \cos(12\, v)
\end{aligned}
$$

$$
\begin{aligned}
& - 1904336\, v^8 \sin(4\, v) + 61776 \sin(13\, v) v^2 + 5700\, v^5 \cos(15\, v) + 3672\, v \cos(6\, v) \\
& + 111432 \cos(10\, v) v^7 + 1485216 \cos(v) v + 1602\, v^4 \sin(6\, v) + 6197124\, v^4 \sin(4\, v) \\
& + 2037240\, v^3 \cos(5\, v) - 33696 \sin(12\, v) - 8572672 \cos(4\, v) v^5 - 12960 \sin(14\, v) \\
& + 142560 \sin(6\, v) - 2695224 \cos(6\, v) v^3 + 353148\, v^7 \cos(5\, v) + 25920 \sin(11\, v) \\
& - 2344248 \cos(2\, v) v^7 - 570220\, v^5 \cos(7\, v) - 2927700 \cos(v) v^7 + 28248 \cos(10\, v) v^3 \\
& - 4512 \sin(13\, v) v^4 + 1372176\, v^2 \sin(v) - 613344 \cos(4\, v) v^3 + 2253192\, v^3 \cos(7\, v) \\
& + 1328832 \cos(4\, v) v + 11515298\, v^6 \sin(2\, v) - 16560\, v^2 \sin(15\, v) - 16416\, v \cos(15\, v) \\
& + 1244444\, v^5 \cos(8\, v) + 27648 \cos(14\, v) v + 1838304 \sin(7\, v) v^4 + 11201604\, v^4 \sin(2\, v) \\
& - 116768\, v^9 \cos(5\, v) - 77760 \sin(9\, v) - 181440 \sin(7\, v) + 4192692\, v^5 \cos(3\, v) \\
& + 406224 \sin(11\, v) v^2 - 73452\, v^7 \cos(11\, v) - 1502760\, v^3 \cos(8\, v) - 31072\, v^6 \sin(13\, v) \\
& + 652896\, v^4 \sin(9\, v) - 650856 \cos(8\, v) v^7 - 97356\, v^4 \sin(10\, v) - 3033216 \cos(4\, v) v^7 \\
& - 3711432 \cos(v) v^3 - 2191752\, v^2 \sin(2\, v) + 754284\, v^8 \sin(6\, v) - 962988\, v^7 \cos(7\, v) \\
& - 939960\, v^3 \cos(3\, v) + 25920 \sin(13\, v) + 3037812 \cos(2\, v) v^5 + 4500\, v^4 \sin(14\, v) \\
& - 32\, v^9 \cos(13\, v) - 891\, v^5 \cos(6\, v) - 311904 \cos(8\, v) v - 304128\, v \cos(10\, v) \\
& - 89832\, v^3 \cos(13\, v) + 336960 \sin(3\, v) - 189216 \sin(4\, v) - 1362660\, v^4 \sin(8\, v) \\
& + 216\, v^7 \cos(14\, v) + 167520 \cos(12\, v) v^3 + 26058\, v^6 \sin(12\, v)
\end{aligned}
$$

$$
\begin{aligned}
F_{21} = & -540\, v^8 \sin(14\, v) + 6176\, v^8 \sin(12\, v) + 177984 \cos(12\, v) v^5 - 129024 \cos(10\, v) v^5 \\
& + 2391\, v^9 \cos(11\, v) + 6047501 \sin(v) v^8 - 2592\, v^3 \cos(11\, v) + 106272\, v^3 \cos(9\, v) \\
& - 207360\, v^3 + 1784493 \sin(3\, v) v^8 - 4651052\, v^8 \sin(2\, v) + 12\, v^{10} \sin(14\, v) \\
& - 7488\, v^5 \cos(13\, v) - 72576\, v^4 \sin(12\, v) + 10368\, v^3 \cos(14\, v) - 2656365\, v^9 \cos(3\, v) \\
& + 215472\, v^{10} \sin(4\, v) - 530544 \cos(6\, v) v^9 + 1022112 \cos(4\, v) v^9 + 15488\, v^{11} \cos(2\, v) \\
& + 38320\, v^{11} \cos(v) - 380004\, v^{10} \sin(2\, v) - 77004\, v^{10} \sin(5\, v) - 320388\, v^{10} \sin(3\, v) \\
& - 3106944\, v^5 - 2592\, v^3 \cos(15\, v) - 458154\, v^7 \cos(9\, v) - 51840 \cos(2\, v) v^3 \\
& + 497664 \cos(6\, v) v^5 - 68292\, v^{11} \cos(3\, v) - 96072 \sin(11\, v) v^6 + 7116\, v^{10} \sin(10\, v) \\
& + 5650344 \sin(3\, v) v^6 + 907925 \sin(7\, v) v^8 - 44391 \sin(9\, v) v^8 + 505392 \cos(2\, v) v^9 \\
& - 4345344\, v^7 - 593280\, v^8 \sin(8\, v) + 186192 \sin(11\, v) v^4 - 144000\, v^6 \sin(8\, v) \\
& - 7439706\, v^7 \cos(3\, v) + 16112\, v^{11} \cos(4\, v) - 804744\, v^7 \cos(6\, v) + 3827328\, v^6 \sin(6\, v) \\
& + 1042416 \sin(9\, v) v^6 - 1035456\, v^9 - 10264\, v^{11} \cos(7\, v) + 81\, v^8 \sin(15\, v) \\
& + 576\, v^6 \sin(15\, v) + 4128\, v^{11} \cos(8\, v) + 7893600 \sin(v) v^6 + 16\, v^{11} \cos(12\, v) \\
& - 605184\, v^6 \sin(10\, v) - 1039 \sin(13\, v) v^8 + 1011120\, v^{10} \sin(v) - 18432\, v^5 \cos(14\, v) \\
& - 4752\, v^4 \sin(15\, v) - 528\, v^{10} \sin(12\, v) + 17088 \cos(8\, v) v^9 - 2945376\, v^5 \cos(5\, v) \\
& - 29055\, v^9 \cos(9\, v) - 87264\, v^6 \sin(4\, v) + 1197504\, v^4 \sin(6\, v) - 3310219\, v^8 \sin(5\, v) \\
& - 647\, v^8 \sin(11\, v) - 1351920\, v^6 \sin(7\, v) + 25008\, v^9 \cos(10\, v) + 40132\, v^8 \sin(10\, v) \\
& - 3744 \cos(12\, v) v^9 + 144\, v^9 \cos(14\, v) + 162\, v^7 \cos(15\, v) - 6336\, v^6 \sin(14\, v) \\
& + 1734480 \sin(3\, v) v^4 + 16476\, v^{10} \sin(6\, v) - 750\, v^7 \cos(13\, v) - 15040\, v^{11} \cos(6\, v) \\
& + 4940208 \cos(v) v^5 - 942192 \sin(5\, v) v^4 - 4726920 \sin(5\, v) v^6 - 6717\, v^9 \cos(7\, v) \\
& + 1773360 \sin(v) v^4 + 699408\, v^5 \cos(9\, v) - 234720\, v^5 \cos(11\, v) - 20256\, v^{11} \\
& + 1804677 \cos(v) v^9 - 27072\, v^7 \cos(12\, v) + 1069728\, v^8 \sin(4\, v) + 3024\, v^5 \cos(15\, v) \\
& + 366312 \cos(10\, v) v^7 - 362880\, v^4 \sin(4\, v) - 12024\, v^{10} \sin(9\, v) + 61752\, v^{10} \sin(7\, v) \\
& + 1044\, v^{10} \sin(11\, v) - 256608\, v^3 \cos(5\, v) - 33216\, v^{10} \sin(8\, v) + 4439232 \cos(4\, v) v^5 \\
& + 4\, v^{11} \cos(13\, v) + 93312 \cos(6\, v) v^3 - 1662750\, v^7 \cos(5\, v) - 124\, v^{11} \cos(11\, v) \\
& + 438312 \cos(2\, v) v^7 + 1578096\, v^5 \cos(7\, v) - 448\, v^{11} \cos(10\, v) + 7139670 \cos(v) v^7
\end{aligned}
$$

$$- 51840 \cos(10 v)v^3 - 12528 \sin(13 v)v^4 + 311040 \cos(4 v)v^3 + 69984 v^3 \cos(7 v)$$
$$- 8411712 v^6 \sin(2 v) - 36 v^{10} \sin(13 v) - 1510272 v^5 \cos(8 v) - 9 v^9 \cos(15 v)$$
$$- 872208 \sin(7 v)v^4 - 2237760 v^4 \sin(2 v) + 884997 v^9 \cos(5 v) - 4033152 v^5 \cos(3 v)$$
$$+ 14694 v^7 \cos(11 v) - 124416 v^3 \cos(8 v) - 6168 v^6 \sin(13 v) + 225072 v^4 \sin(9 v)$$
$$+ 38844 v^{11} \cos(5 v) - 1179648 \cos(8 v)v^7 - 302400 v^4 \sin(10 v) + 5552064 \cos(4 v)v^7$$
$$+ 324000 \cos(v)v^3 + 1419188 v^8 \sin(6 v) + 2406834 v^7 \cos(7 v) - 220320 v^3 \cos(3 v)$$
$$- 350208 \cos(2 v)v^5 + 22464 v^4 \sin(14 v) + 81 v^9 \cos(13 v) - 18144 v^3 \cos(13 v)$$
$$+ 1512 v^{11} \cos(9 v) + 290304 v^4 \sin(8 v) + 120 v^7 \cos(14 v) + 20736 \cos(12 v)v^3$$
$$+ 125088 v^6 \sin(12 v)$$

$$F_{22} = -4788288 v - 2718520 \sin(3 v)v^8 - 6660764 v^6 \sin(8 v)+145152 \sin(10 v) + 741312 \sin(8 v)$$
$$+ 528768 \sin(6 v) - 466560 \sin(9 v) - 798336 \sin(7 v)+51840 \sin(11 v) - 275180 v^8 \sin(10 v)$$
$$- 990144 \sin(4 v) + 1036800 \sin(v)x + 176256 \sin(5 v) - 18 v^7 \cos(6 v) - 3022528 \sin(v)v^8$$
$$- 1296 \sin(18 v) + 157736 v^8 \sin(11 v) - 5102400 v^4 \sin(10 v) + 28287418 v^6 \sin(2 v)$$
$$- 13712256 v^3 \cos(8 v) - 3158716 v^8 \sin(6 v) + 9 v^7 \cos(18 v) - 39717 v^5 \cos(14 v)$$
$$- 44992448 \cos(v)v^5 - 4353 v^6 \sin(14 v) - 3648 \cos(12 v)v^9 + 46656 \sin(15 v)$$
$$+ 5184 \sin(17) + 749688 v^9 \cos(3 v) - 13477824 \cos(6 v)v^3 - 139968 \sin(12 v) + 7474474 v^7$$
$$- 1135296 \cos(4 v)v^9 + 596 v^5 \cos(6 v) + 8 v^{10} \sin(14 v) + 698688 v^2 \sin(9 v)$$
$$- 1002240 \cos(8 v)v + 7255104 v^3 \cos(9 v) - 1660512 \sin(11 v)v^6 + 660728 v^7 \cos(11 v)$$
$$- 287536 \cos(8 v)v^9 - 1048516 \cos(10 v)v^7 - 1447 v^7 \cos(14 v) - 22120 v^5 \cos(15 v)$$
$$- 23552 v^{10} \sin(8 v) + 982980 v^4 \sin(12 v) - 288 v^3 \cos(18) - 2172 v^4 \sin(17)$$
$$- 1608768 v \cos(10 v) - 613752 \cos(v)v^9 + 79232 v^7 \cos(12 v) + 1769472 v \cos(9 v)$$
$$+ 41714292 v^4 \sin(4 v) - 14504 v^6 \sin(15 v) - 3703888 v^5 \cos(9 v) + 31487218 \cos(2 v)v^5$$
$$+ 510960 v^5 \cos(13 v) + 1602552 \sin(7 v)v^4 + 1531884 v^8 \sin(8 v) - 2759040 v^2 \sin(10 v)$$
$$- 11712852 v^4 \sin(8 v) - 297216 \cos(12 v)v - 405 v^5 \cos(18) + 3272 v^{10} \sin(10 v)$$
$$+ 5240536 v^7 \cos(3 v) - 40 v^9 \cos(15 v) - 5391936 \sin(5 v)v^2 - 7720624 v^5 \cos(3 v)$$
$$+ 14275200 \cos(2 v)v^3 - 26953248 \sin(3 v)v^6 + 692 v^8 \sin(15 v) + 31248 v^2 \sin(6 v)$$
$$- 57384 v^2 \sin(14 v) - 9696 v^3 \cos(6 v) + 39744 v \cos(6 v) - 1211616 v^2 \sin(4 v)$$
$$+ 98216 v^{10} \sin(6 v) - 13536 v^2 \sin(17) + 3024 v \cos(18) - 12021248 \cos(4 v)v^7$$
$$- 29606808 \sin(5 v)v^4 - 10351216 v^6 \sin(6 v) - 168 v^7 \cos(17) - 14648832 \sin(v)v^4$$
$$+ 26693046 v^4 \sin(2 v) + 4320 v^3 \cos(17) - 975936 \cos(10 v)v^3 + 9482880 \cos(4 v)v^3$$
$$+ 596160 \sin(13 v)v^2 - 743336 \sin(9 v)v^8 - 3575136 \sin(5 v)v^6 + 10083192 v^4 \sin(9 v)$$
$$+ 15867840 v^3 \cos(7 v) - 6890688 \sin(7 v)v^2 + 2698944 v^3 \cos(5 v) - 99 v^6 \sin(18)$$
$$- 1926808 v^7 \cos(9 v) + 3732614 \cos(2 v)v^7 - 193428 v^6 \sin(12 v) - 780 v^8 \sin(14 v)$$
$$+ 22340 v^8 \sin(12 v) + 2376 v^2 \sin(18) + 926608 \cos(6 v)v^9 - 256 v^{10} \sin(12 v)$$
$$+ 5155876 v^8 \sin(2 v) + 8505216 v^2 \sin(v) - 38118360 \sin(3 v)v^4 + 4467560 \cos(8 v)v^7$$
$$- 9798144 \cos(v)v^3 + 144 v^9 \cos(14 v) + 133864 v^9 \cos(9 v) - 2455600 v^5 \cos(11 v)$$
$$+ 2472 v^5 \cos(17) - 966 v^4 \sin(6 v) + 850648 \sin(7 v)v^8 + 19405308 v^6 \sin(4 v)$$
$$+ 25055260 v^5 + 2589024 v^3 - 1515016 v^7 \cos(7 v) + 640052 v^9 - 354120 v^9 \cos(7 v)$$
$$+ 8286952 v^7 \cos(5 v) + 24807152 v^5 \cos(5 v) + 102984 v^9 \cos(5 v) - 280076 v^8 \sin(4 v)$$
$$+ 2070664 v^8 \sin(5 v) - 14232 \sin(13 v)v^8 + 9218880 v^2 \sin(6 v) + 134784 \sin(13 v)$$
$$- 13824 v \cos(13 v) - 96768 v \cos(15 v) - 19944 v^9 \cos(11 v) + 1320 v^9 \cos(13 v)$$
$$+ 44464 v^9 \cos(10 v) - 12 v^8 \sin(17) + 26 v^8 \sin(6 v) - 6068736 v \cos(5 v)$$

$$
\begin{aligned}
&+ 8874432\,\sin(3\,v)v^2 - 2419200\,v\cos(3\,v) + 113616\,\cos(14\,v)v + 178848\,v^3\cos(14\,v)\\
&- 16498320\,\sin(v)v^6 - 2682660\,v^7\cos(6\,v) + 6048000\,\cos(4\,v)v + 639\,v^4\sin(18)\\
&- 1692576\,\sin(2\,v) - 18144\,\sin(6\,v) - 4\,v^9\cos(6\,v) - 1265088\,v^3\cos(11\,v)\\
&- 1000512\,v^3\cos(13\,v) - 857016\,\sin(13\,v)v^4 + 2649312\,v^2\sin(8\,v) - 13778112\,v^3\cos(3\,v)\\
&+ 14699728\,v^5\cos(7\,v) - 1382400\,v\cos(7\,v) + 9152\,v^{10}\sin(9\,v) - 38336\,v^{10}\sin(7\,v)\\
&- 928\,v^{10}\sin(11\,v) + 7243776\,\cos(v)v + 981504\,v\cos(11\,v) + 382\,v^6\sin(6\,v)\\
&+ 13572\,v^4\sin(15\,v) - 1055856\,v^5\cos(8\,v) + 4560\,v^7\cos(15\,v) - 214272\,v^{10}\sin(4\,v)\\
&- 430000\,\cos(12\,v)v^5 + 4982132\,\cos(10\,v)v^5 + 15648\,v^3\cos(15\,v) - 26992044\,\cos(6\,v)v^5\\
&- 86832\,\sin(14\,v) + 1470600\,\sin(11\,v)v^4 + 4114368\,\cos(6\,v)v + 32\,v^{10}\sin(13\,v)\\
&+ 936\,v^6\sin(17) - 184784\,\cos(2\,v)v^9 - 112000\,v^{10}\sin(v) - 1404000\,v^2\sin(12\,v)\\
&- 73912\,v^7\cos(13\,v) - 13481136\,v^2\sin(2\,v) - 65376\,v^2\sin(15\,v) + 10358496\,v^6\sin(7\,v)\\
&- 14132816\,\cos(4\,v)v^5 + 1650048\,\cos(12\,v)v^3 + 213224\,v^{10}\sin(2\,v) + 62304\,v^{10}\sin(5\,v)\\
&- 1248\,v^{10}\sin(3\,v) + 13468200\,v^4\sin(6\,v) - 10676872\,\cos(v)v^7 + 1571104\,\sin(9\,v)v^6\\
&- 2622240\,\cos(2\,v)v + 231648\,v^6\sin(13\,v) - 13824\,v\cos(17) + 1586304\,\sin(3\,v)\\
&+ 125133\,v^4\sin(14\,v) + 2761528\,v^6\sin(10\,v) + 2476224\,\sin(11\,v)v^2
\end{aligned}
$$

$$
\begin{aligned}
F_{23} = &\ 12276400\,\sin(3\,v)v^8 - 259680\,v^6\sin(8\,v) + 562924\,v^8\sin(10\,v) + 196852\,v^{11}\cos(5\,v)\\
&+ 252\,v^7\cos(6\,v) + 18431104\,\sin(v)v^8 - 486000\,v^8\sin(11\,v) - 931392\,v^4\sin(10\,v)\\
&- 36061248\,v^6\sin(2\,v) - 352512\,v^3\cos(8\,v) + 8874172\,v^8\sin(6\,v) + 1152\,v^5\cos(14\,v)\\
&+ 15365952\,\cos(v)v^5 + 21696\,v^6\sin(14\,v) - 4552\,\cos(12\,v)v^9 - 8775788\,v^9\cos(3\,v)\\
&+ 373248\,\cos(6\,v)v^3 - 22172940\,v^7 + 8637448\,\cos(4\,v)v^9 - 10800\,v^5\cos(6\,v)\\
&+ 1320\,v^{11}\cos(12\,v) - 18\,v^{10}\sin(14\,v) + 352512\,v^3\cos(9\,v) + 1267200\,\sin(11\,v)v^6\\
&- 1408752\,v^7\cos(11\,v) + 1023860\,\cos(8\,v)v^9 + 2025408\,\cos(10\,v)v^7 - 5568\,v^7\cos(14\,v)\\
&+ 59328\,v^5\cos(15\,v) - 121794\,v^{10}\sin(8\,v) - 195916\,v^{11}\cos(6\,v) - 362880\,v^4\sin(12\,v)\\
&- 106612\,v^{12}\sin(2\,v) + 7782092\,\cos(v)v^9 - 117792\,v^7\cos(12\,v) - 1620864\,v^4\sin(4\,v)\\
&+ 31104\,v^6\sin(15\,v) - 4\,v^{12}\sin(14\,v) + 3871296\,v^5\cos(9\,v) - 1775232\,\cos(2\,v)v^5\\
&- 414144\,v^5\cos(13\,v) - 3193344\,\sin(7\,v)v^4 - 4592056\,v^8\sin(8\,v) + 1330560\,v^4\sin(8\,v)\\
&+ 56000\,v^{12}\sin(v) + 53406\,v^{10}\sin(10\,v) - 1636\,v^{12}\sin(10\,v) - 20305200\,v^7\cos(3\,v)\\
&- 828\,v^9\cos(15\,v) - 10052928\,v^5\cos(3\,v) - 207360\,\cos(2\,v)v^3 - 44\,v^{11}\cos(14\,v)\\
&+ 21787392\,\sin(3\,v)v^6 + 960\,v^8\sin(15\,v) + 1391452\,v^{11}\cos(2\,v) + 5184\,v^3\cos(6\,v)\\
&- 861210\,v^{10}\sin(6\,v) - 1087760\,v^{11} + 28582176\,\cos(4\,v)v^7\\
&- 1886976\,\sin(5\,v)v^4 - 572\,v^{11}\cos(13\,v) + 16681152\,v^6\sin(6\,v) + 4838400\,\sin(v)v^4\\
&- 7560000\,v^4\sin(2\,v) + 11776\,v^{12}\sin(8\,v) - 207360\,\cos(10\,v)v^3 + 953856\,\cos(4\,v)v^3\\
&- 4576\,v^{12}\sin(9\,v) + 19168\,v^{12}\sin(7\,v) + 2652256\,\sin(9\,v)v^8 - 11421696\,\sin(5\,v)v^6\\
&+ 96768\,v^4\sin(9\,v) + 62208\,v^3\cos(7\,v) - 808704\,v^3\cos(5\,v) + 2087856\,v^7\cos(9\,v)\\
&+ 2843328\,\cos(2\,v)v^7 + 431712\,v^6\sin(12\,v) + 71516\,v^{11}\cos(7\,v) + 2828\,v^8\sin(14\,v)\\
&- 12072\,v^8\sin(12\,v) - 5974704\,\cos(6\,v)v^9 - 4326\,v^{10}\sin(12\,v) - 30243364\,v^8\sin(2\,v)\\
&- 184488\,v^{11}\cos(4\,v) + 5564160\,\sin(3\,v)v^4 - 6291696\,\cos(8\,v)v^7 + 933120\,\cos(v)v^3\\
&- 1264\,v^9\cos(14\,v) - 31152\,v^{12}\sin(5\,v) - 717604\,v^9\cos(9\,v) - 327744\,v^5\cos(11\,v)\\
&+ 12096\,v^4\sin(6\,v) + 128\,v^{12}\sin(12\,v) - 2026720\,\sin(7\,v)v^8 - 759072\,v^6\sin(4\,v)\\
&- 11653776\,v^5 - 648000\,v^3 + 7940112\,v^7\cos(7\,v) - 9656855\,v^9 + 3039380\,v^9\cos(7\,v)\\
&- 13409232\,v^7\cos(5\,v) - 11434176\,v^5\cos(5\,v) - 1332596\,v^9\cos(5\,v) + 624\,v^{12}\sin(3\,v)\\
&+ 8828408\,v^8\sin(4\,v) - 11334640\,v^8\sin(5\,v) + 15920\,\sin(13\,v)v^8 - 15844\,v^{11}\cos(10\,v)
\end{aligned}
$$

$$
\begin{aligned}
&- 8572\, v^9 \cos(11\, v) + 13916\, v^9 \cos(13\, v) + 9584\, v^9 \cos(10\, v) - 324\, v^8 \sin(6\, v)\\
&- 49108\, v^{12} \sin(6\, v) + 464\, v^{12} \sin(11\, v) + 41472\, v^3 \cos(14\, v) + 22687104\, \sin(v) v^6\\
&- 4863168\, v^7 \cos(6\, v) + 107136\, v^{12} \sin(4\, v) + 99\, v^9 \cos(6\, v) + 62208\, v^3 \cos(11\, v)\\
&- 62208\, v^3 \cos(13\, v) + 91280\, v^{11} \cos(8\, v) - 658676\, v^{11} \cos(3\, v) + 89856\, \sin(13\, v) v^4\\
&+ 9\, v^{10} \sin(6\, v) - 518400\, v^3 \cos(3\, v) + 2932416\, v^5 \cos(7\, v) + 60120\, v^{10} \sin(9\, v)\\
&+ 581112\, v^{10} \sin(7\, v) - 41928\, v^{10} \sin(11\, v) - 4176\, v^6 \sin(6\, v) - 55296\, v^4 \sin(15\, v)\\
&+ 430052\, v^{11} \cos(v) - 5652288\, v^5 \cos(8\, v) - 6576\, v^7 \cos(15\, v) + 4336146\, v^{10} \sin(4\, v)\\
&+ 683136\, \cos(12\, v) v^5 - 890496\, \cos(10\, v) v^5 - 20736\, v^3 \cos(15\, v) - 47692\, v^{11} \cos(9\, v)\\
&+ 2664576\, \cos(6\, v) v^5 + 822528\, \sin(11\, v) v^4 + 4920\, v^{10} \sin(13\, v) + 5966384\, \cos(2\, v) v^9\\
&+ 3336312\, v^{10} \sin(v) + 185328\, v^7 \cos(13\, v) - 9374592\, v^6 \sin(7\, v) + 16633728\, \cos(4\, v) v^5\\
&- 168\, v^{10} \sin(15\, v) + 41472\, \cos(12\, v) v^3 + 8508\, v^{11} \cos(11\, v) - 5842506\, v^{10} \sin(2\, v)\\
&- 2016552\, v^{10} \sin(5\, v) + 845784\, v^{10} \sin(3\, v) + 12\, v^{11} \cos(15\, v) + 3955392\, v^4 \sin(6\, v)\\
&+ 24916464\, \cos(v) v^7 + 2879616\, \sin(9\, v) v^6 - 433920\, v^6 \sin(13\, v) + 50112\, v^4 \sin(14\, v)\\
&- 2826816\, v^6 \sin(10\, v) - 16\, v^{12} \sin(13\, v)
\end{aligned}
$$

For some values of $|w|$, the formulae given by (26) may be subject to heavy cancelations. In this case the following Taylor series expansions should be used:

$$
\begin{aligned}
a_2 &= -2 - \frac{19\, v^8}{12096} - \frac{83\, v^{10}}{453600} - \frac{127\, v^{12}}{2661120} - \frac{51691\, v^{14}}{6538371840}\\
&\quad - \frac{10032119\, v^{16}}{6276836966400} - \frac{205410173\, v^{18}}{666913927680000} + \cdots\\
b_4 &= \frac{3}{40} + \frac{19\, v^2}{1512} + \frac{451\, v^4}{181440} + \frac{241\, v^6}{498960} + \frac{2039267\, v^8}{21794572800} + \frac{5926819\, v^{10}}{326918592000}\\
&\quad + \frac{2371353869\, v^{12}}{666913927680000} + \frac{281162192833\, v^{14}}{399147985716480000} + \frac{147966204882971\, v^{16}}{1053750682291507200000}\\
&\quad + \frac{6312228847417\, v^{18}}{225104634916761600000} + \cdots\\
b_3 &= \frac{13}{15} - \frac{19\, v^2}{378} + \frac{101\, v^4}{11340} + \frac{71\, v^6}{35640} + \frac{1689913\, v^8}{5448643200} + \frac{2024909\, v^{10}}{32691859200}\\
&\quad + \frac{228936707\, v^{12}}{20841060240000} + \frac{217490976251\, v^{14}}{99786996429120000} + \frac{35070479772577\, v^{16}}{81057744791654400000}\\
&\quad + \frac{979646901088223\, v^{18}}{11185968781248307200000} + \cdots\\
b_2 &= \frac{7}{60} + \frac{19\, v^2}{252} - \frac{689\, v^4}{30240} + \frac{95\, v^6}{12474} - \frac{583931\, v^8}{1210809600} + \frac{1687369\, v^{10}}{18162144000}\\
&\quad - \frac{6035387\, v^{12}}{8550178560000} + \frac{94581997171\, v^{14}}{6652466428608000} + \frac{30528574110311\, v^{16}}{1756251137152512000000}\\
&\quad + \frac{573876496041179\, v^{18}}{12118132846352332800000} + \cdots
\end{aligned}
\qquad (27)
$$

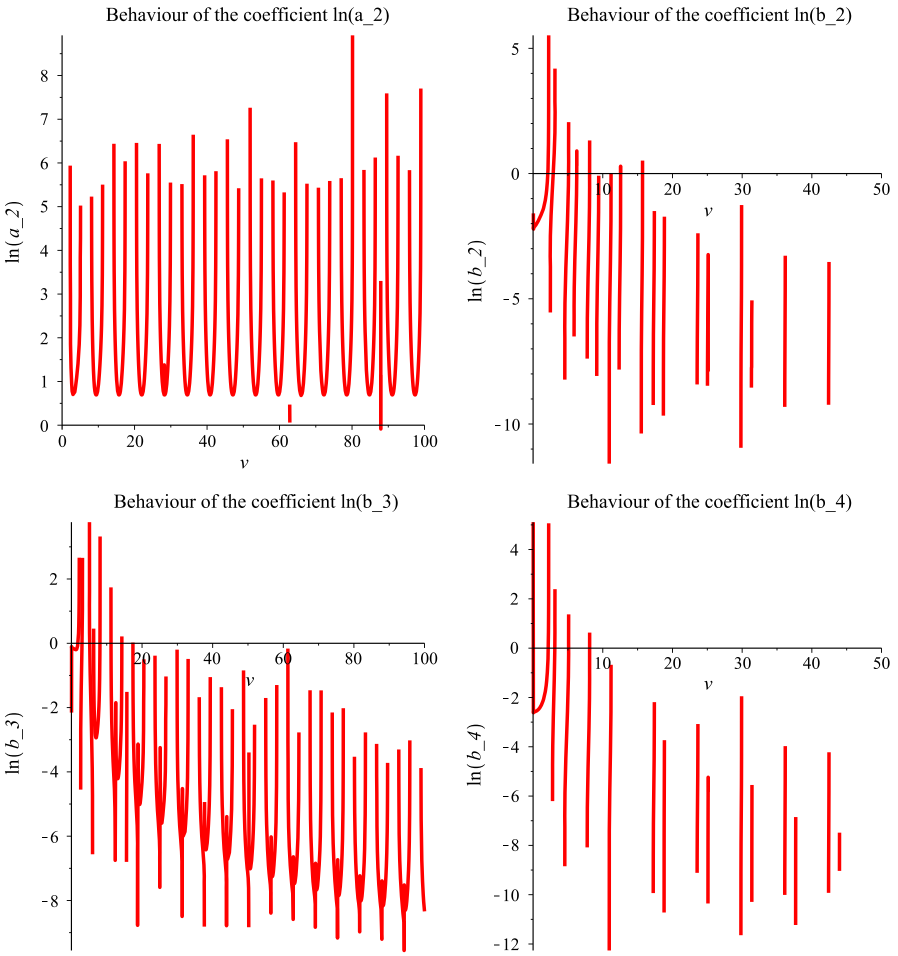


Fig. 5 Behavior of the coefficients of the new proposed method given by (26) for several values of $v = w\,h$

Figure 5 shows the behavior of the coefficients $a_2\, b_2,\ b_3$ and b_4.

In order now to have the proposed method (11), a combination of the above first block level (with two stages) and second level (with the third stage) must be hold. The above mentioned combination leads to the proposed method (11) with the coefficients given by (18)–(19) and (26)–(27).

The local truncation error of this new proposed method (mentioned as $HybTyp\,Meth$) is given by:

$$LTE_{HybTypMeth} = -\frac{19\,h^8}{6048}\left(p_n^{(8)} + 4\,w^2\,p_n^{(6)} + 6w^4\,p_n^{(4)} + 4\,w^6\,p_n^{(2)} + w^8\,p_n\right) + O\left(h^{10}\right) \tag{28}$$

where $p_n^{(j)}$ is the jth derivative of p_n.

4 Comparative error analysis

In this chapter, we will investigate the following methods:

4.1 Classical method [i.e. the method (11) with constant coefficients]

$$LTE_{CL} = -\frac{19\,h^8}{6048}\,p_n^{(8)} + O\left(h^{10}\right) \tag{29}$$

4.2 The hybrid method with vanished phase-lag and its first derivative in each level developed in [40]

$$LTE_{Meth\,I} = \frac{751\,h^8}{302400}\,\left(p_n^{(8)} + 2\,w^2\,p_n^{(6)} + w^4\,p_n^{(4)}\right) + O\left(h^{10}\right) \tag{30}$$

4.3 Runge–Kutta type method with vanished phase-lag and its first and second derivatives in each level developed in [41]

$$LTE_{Meth\,II} = -\frac{751\,h^8}{302400}\,\left(p_n^{(8)} + 3\,w^2\,p_n^{(6)} + 3\,w^4\,p_n^{(4)} + w^6\,p_n^{(2)}\right) + O\left(h^{10}\right) \tag{31}$$

4.4 Hybrid type method with vanished phase-lag and its first, second and third derivatives in each level or block-level developed in Sect. 3

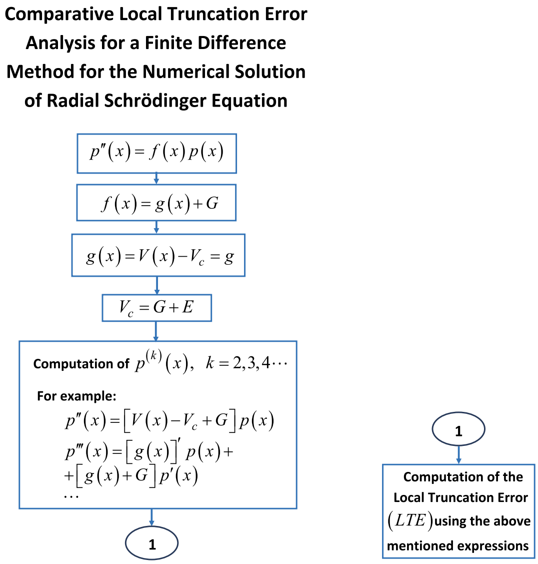
$$\begin{aligned} LTE_{Meth\,III} = & -\frac{19\,h^8}{6048}\,\left(p_n^{(8)} + 4\,w^2\,p_n^{(6)} + 6w^4\,p_n^{(4)} + 4\,w^6\,p_n^{(2)} + w^8\,p_n\right) \\ & + O\left(h^{10}\right) \end{aligned} \tag{32}$$

In order to proceed to the the local truncation error analysis, the flowchart mentioned in the Fig. 6 is followed.

We use the algorithm described on the flowchart together with the formulae:

$$\begin{aligned} p_n^{(2)} &= \left(V(x) - V_c + G\right)\,p(x) \\ p_n^{(3)} &= \left(\frac{d}{dx}g(x)\right)p(x) + \left(g(x) + G\right)\frac{d}{dx}p(x) \\ p_n^{(4)} &= \left(\frac{d^2}{dx^2}g(x)\right)p(x) + 2\left(\frac{d}{dx}g(x)\right)\frac{d}{dx}p(x) \\ &\quad + \left(g(x) + G\right)^2\,p(x) \end{aligned}$$

Fig. 6 Flowchart for the comparative error analysis



$$
\begin{aligned}
p_n^{(5)} &= \left(\frac{d^3}{dx^3}g(x)\right)p(x)+3\left(\frac{d^2}{dx^2}g(x)\right)\frac{d}{dx}p(x)\\
&\quad +4\,(g(x)+G)\,p(x)\frac{d}{dx}g(x)+(g(x)+G)^2\frac{d}{dx}p(x)\\
p_n^{(6)} &= \left(\frac{d^4}{dx^4}g(x)\right)p(x)+4\left(\frac{d^3}{dx^3}g(x)\right)\frac{d}{dx}p(x)\\
&\quad +7\,(g(x)+G)\,p(x)\frac{d^2}{dx^2}g(x)+4\left(\frac{d}{dx}g(x)\right)^2p(x)\\
&\quad +6\,(g(x)+G)\left(\frac{d}{dx}p(x)\right)\frac{d}{dx}g(x)+(g(x)+G)^3\,p(x)\\
p_n^{(7)} &= \left(\frac{d^5}{dx^5}g(x)\right)p(x)+5\left(\frac{d^4}{dx^4}g(x)\right)\frac{d}{dx}p(x)\\
&\quad +11\,(g(x)+G)\,p(x)\frac{d^3}{dx^3}g(x)+15\left(\frac{d}{dx}g(x)\right)p(x)\frac{d^2}{dx^2}g(x)\\
&\quad +13\,(g(x)+G)\left(\frac{d}{dx}p(x)\right)\frac{d^2}{dx^2}g(x)+10\left(\frac{d}{dx}g(x)\right)^2\frac{d}{dx}p(x)\\
&\quad +9\,(g(x)+G)^2\,p(x)\frac{d}{dx}g(x)+(g(x)+G)^3\frac{d}{dx}p(x)\\
p_n^{(8)} &= \left(\frac{d^6}{dx^6}g(x)\right)p(x)+6\left(\frac{d^5}{dx^5}g(x)\right)\frac{d}{dx}p(x)\\
&\quad +16\,(g(x)+G)\,p(x)\frac{d^4}{dx^4}g(x)+26\left(\frac{d}{dx}g(x)\right)p(x)\frac{d^3}{dx^3}g(x)
\end{aligned}
$$

$$+24\ (g(x)+G)\left(\frac{d}{dx}p(x)\right)\frac{d^3}{dx^3}g(x)+15\left(\frac{d^2}{dx^2}g(x)\right)^2 p(x)$$
$$+48\left(\frac{d}{dx}g(x)\right)\left(\frac{d}{dx}p(x)\right)\frac{d^2}{dx^2}g(x)+22\ (g(x)+G)^2\,p(x)\frac{d^2}{dx^2}g(x)$$
$$+28\ (g(x)+G)\,p(x)\left(\frac{d}{dx}g(x)\right)^2$$
$$+12\ (g(x)+G)^2\left(\frac{d}{dx}p(x)\right)\frac{d}{dx}g(x)+(g(x)+G)^4\,p(x)$$
$$\ldots$$

We use the above described methodology in order to obtain the formulae of the Local Truncation Errors.

Investigating the formulae of the Local Truncation Errors, we give attention on two cases in terms of the value of E:

– The Energy is closed to the potential, i.e., $G = V_c - E \approx 0$. Therefore, all the terms of the Local Truncation Error with power of G (i.e. G^j, $j \neq 0$) are approximately equal to zero and we consider only the terms of the polynomials which are free of G. Consequently, for these values of G (i.e. approximately equal to zero), all the the methods are of comparable accuracy. We note that in this case the terms of the polynomials free which are of G are the *same* for the cases of the classical method and of the methods with vanished the phase-lag and its derivatives.
– $G \gg 0$ or $G \ll 0$. Then $|G|$ is a large number.

We have the following asymptotic expansions of the Local Truncation Errors which are based on the analysis presented above:

4.5 Classical method

$$LTE_{CL} = h^8\left(-\frac{19}{6048}\,p\,(x)\;G^4+\cdots\right)+O\left(h^{10}\right) \tag{33}$$

4.6 The hybrid method with vanished phase-lag and its first derivative in each level developed in in [40]

$$LTE_{Meth\,I} = h^8\left[\left(\frac{751}{33600}\left(\frac{d^2}{dx^2}g(x)\right)p(x)+\frac{751}{151200}\left(\frac{d}{dx}g(x)\right)\frac{d}{dx}p(x)\right.\right.$$
$$\left.\left.+\frac{751}{302400}\,(g(x))^2\;p(x)\right)G^2+\cdots\right]+O\left(h^{10}\right) \tag{34}$$

4.7 Runge–Kutta type method with vanished phase-lag and its first and second derivatives in each level developed in [41]

$$LTE_{Meth\,II} = h^8 \left[\left(\frac{751}{75600} \left(\frac{d^2}{dx^2} g(x) \right) p(x) \right) G^2 + \cdots \right] + O\left(h^{10}\right) \quad (35)$$

4.8 Hybrid type method with vanished phase-lag and its first, second and third derivatives in each level or block-level developed in Sect. 3

$$\begin{aligned} LTE_{Meth\,III} = h^8 \Bigg[\Bigg(& -\frac{19 \left(\frac{\mathrm{d}^4}{\mathrm{d}x^4} g(x) \right) p(x)}{504} \\ & - \frac{19 \left(\frac{\mathrm{d}^3}{\mathrm{d}x^3} g(x) \right) \frac{\mathrm{d}}{\mathrm{d}x} p(x)}{756} - \frac{19\, g(x)\, p(x) \frac{\mathrm{d}^2}{\mathrm{d}x^2} g(x)}{378} \\ & - \frac{19 \left(\frac{\mathrm{d}}{\mathrm{d}x} g(x) \right)^2 p(x)}{504} \Bigg) G + \cdots \Bigg] + O\left(h^{10}\right) \end{aligned} \quad (36)$$

From the above equations, we have the following theorem:

Theorem 2 *Based on the above analysis we have proved that:*

- *For the Classical Runge–Kutta type Four-Step Method the error increases as the fourth power of G.*
- *For the method with vanished phase-lag and its first derivative in each level which developed in* [40]*, the error increases as the second power of G.*
- *For the the method with vanished phase-lag and its first and second derivatives in each level developed in* [41]*, the error increases as the second power of G.*
- *For the the method with vanished phase-lag and its first, second and third derivatives in each level developed in Sect. 3, the error increases as the first power of G.*

So, for the numerical solution of the time independent radial Schrödinger equation the new proposed method with vanished phase-lag and its its first, second and third derivatives in each level is the most efficient, from theoretical point of view, especially for large values of $|G| = |V_c - E|$.

5 Stability analysis

The flowchart mentioned in Fig. 7 describes the procedure which must be followed for the stability analysis of a Symmetric four step method using a scalar test equation with frequency different than the frequency of the phase-lag analysis.

Based on the above mentioned procedure, we will investigate the stability of the new obtained Hybrid Type Method as follows:

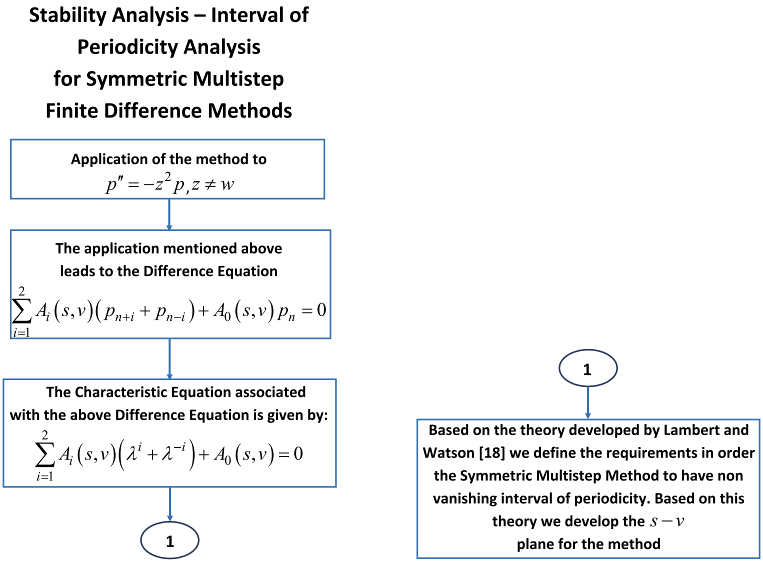


Fig. 7 Flowchart for the stability analysis

– We will investigate the stability of the first block layer (stages 1 and 2) of the new proposed hybrid type method
– We will investigate the stability of the second layer (stage 3) of the new proposed hybrid type method
– We will investigate the stability of the new proposed hybrid type three stages method

5.1 Stability analysis for the first block layer of the new proposed hybrid type method

We consider the first block layer (stages 1 and 2) of the new proposed hybrid type method (12) with the coefficients given by (18).

Applying the above mentioned block layer to the scalar test equation:

$$p'' = -z^2\, p \tag{37}$$

we obtain the following difference equation:

$$A_2(s,v)\ (p_{n+2} + p_{n-2}) + A_1(s,v)\ (p_{n+1} + p_{n-1}) + A_0(s,v)\ p_n = 0 \tag{38}$$

where

$$\begin{aligned} A_2(s,v) = 1,\ A_1(s,v)\ &= a_1 + s^2\left(s^2\, a_0\, b_1 + b_0\right)\ A_0(s,v) \\ &= 2 + s^2\left(-2\, s^2\, a_0 + 1\right)\ b_1 \end{aligned} \tag{39}$$

where $s = z\,h$.

Remark 4 The frequency of the scalar test Eq. (6), w, is not equal with the frequency of the scalar test Eq. (37), z, i.e. $z \neq w$.

The characteristic equation corresponding to the difference Eq. (38) is given by:

$$A_2(s, v)\,\left(\lambda^4 + 1\right) + A_1(s, v)\,\left(\lambda^3 + \lambda\right) + A_0(s, v)\,\lambda^2 = 0 \tag{40}$$

Definition 1 (See [18]) A symmetric $2\,k$-step method with the characteristic equation given by (8) is said to have an *interval of periodicity* $\left(0, v_0^2\right)$ if, for all $s \in \left(0, s_0^2\right)$, the roots $\lambda_i,\ i = 1(1)4$ satisfy

$$\lambda_{1,2} = e^{\pm i\,\zeta(s)},\ |\lambda_i| \leq 1,\ i = 3, 4, \cdots \tag{41}$$

where $\zeta(s)$ is a real function of $z\,h$ and $s = z\,h$.

Definition 2 (See [18]) A method is called P-stable if its interval of periodicity is equal to $(0, \infty)$.

Definition 3 A method is called singularly almost P-stable if its interval of periodicity is equal to $(0, \infty) - S$[1] only when the frequency of the phase fitting is the same as the frequency of the scalar test equation, i.e. $s = v$.

In Fig. 8, we present the $s - v$ plane for the first block layer (stages 1 and 2) of the Hybrid type method developed in this paper. The stable area of the the $s - v$ region of the first block layer (stages 1 and 2) is the shadowed area, while the unstable area is the white area.

5.2 Stability analysis for the second layer (stage 3) of the new proposed hybrid type method

Applying now the second layer (stage 3) of the new produced hybrid type method (20) with the coefficients given by (26) to the scalar test Eq. (37). This leads to the difference Eq. (38) with:

$$A_2(s, v) = 1 + s^2\,b_4\ A_1(s, v) = a_2 + s^2\,b_3\ A_0(s, v) = 2 + s^2\,b_2 \tag{42}$$

where $s = z\,h$.

In Fig. 9, we present the $s{-}v$ plane for the second layer (stage 3) of the hybrid type method developed in this paper. A shadowed area denotes the $s{-}v$ region where the method is stable, while a white area denotes the region where the method is unstable.

[1] Where S is a set of distinct points.

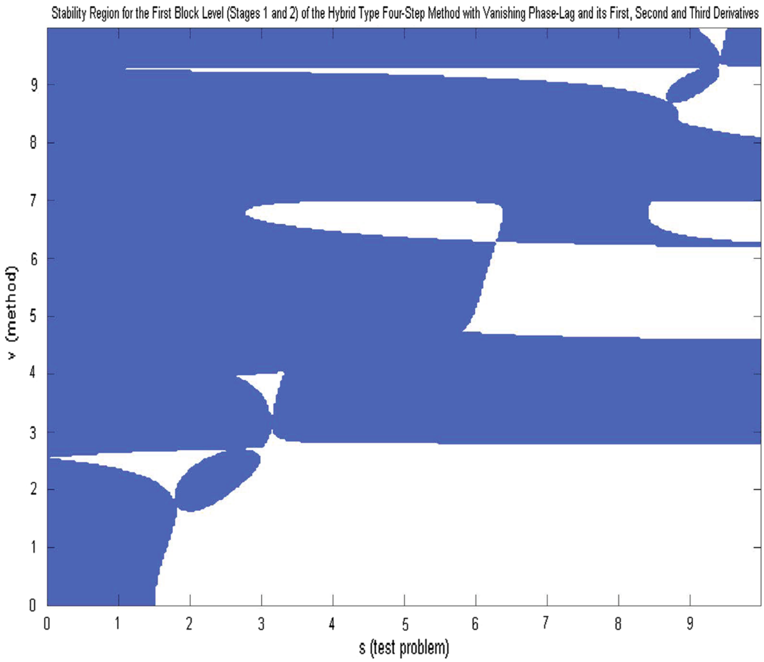


Fig. 8 $s - v$ plane of the first block layer (stages 1 and 2) of the new hybrid type obtained method

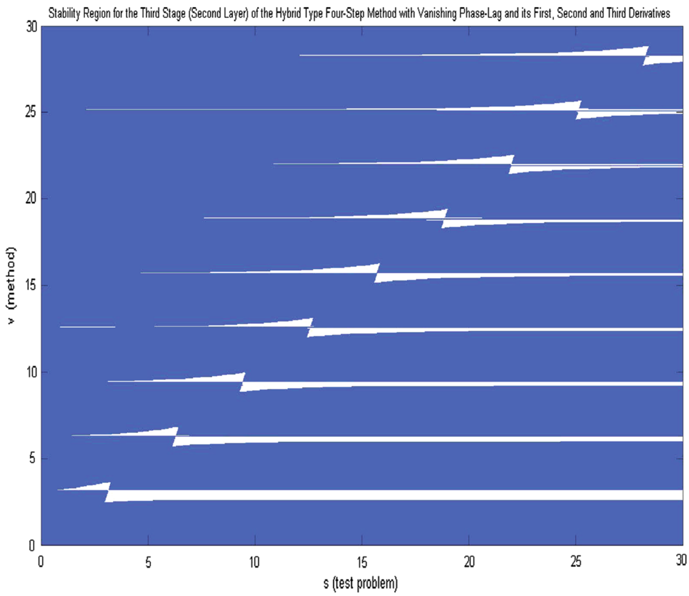


Fig. 9 $s-v$ plane of the second layer (stage 3) of the new Hybrid type obtained method

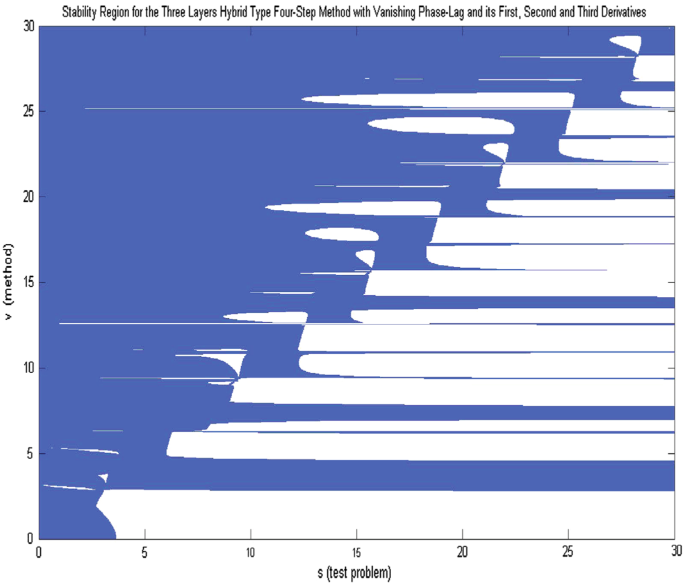


Fig. 10 $s-v$ plane of the new Hybrid type obtained method

5.3 Stability analysis for the new proposed Runge–Kutta type method

Let us apply the new obtained method (11) with the coefficients given by (18)–(19) and (26)–(27) to the scalar test Eq. (37). This leads to the difference Eq. (38) with:

$$
\begin{aligned}
A_2(s, v) &= 1, \\
A_1(s, v) &= a_2 + s^2\ (b_3 - a_1\, b_4) - s^4\, b_0\, b_4 - s^6\, a_0\, b_1\, b_4 \\
A_0(s, v) &= 2 + s^2\ (b_2 - 2\, a_1\, b_4) - s^4\, b_1\, b_4 + 2\, s^6\, a_0\, b_1\, b_4
\end{aligned} \tag{43}
$$

and $s = z\,h$.

In Fig. 10, we present the $s-v$ plane for the hybrid type method developed in this paper. A shadowed area denotes the $s-v$ region where the method is stable, while a white area denotes the region where the method is unstable.

Remark 5 For some problems, as for example the Schrödinger equation and related problems, the frequency of the scalar test equation for the phase-lag analysis is equal to the frequency of the scalar test equation for the stability analysis. So, for this case of problems it is necessary to observe *the surroundings of the first diagonal of the $s-v$ plane*.

We study now the case where the frequency of the scalar test equation for the phase-lag analysis is equal to the frequency of the scalar test equation for the stability analysis, i.e. in the case that $s = v$ (i.e. see the surroundings of the first diagonal of the $s-v$ plane). Based on this study, we extract the result that the interval of periodicity of the new method developed in Sect. 3 is equal to: (0, 434.1806).

From the above analysis we have the following theorem:

Theorem 3 *The method developed in Sect. 3:*

- *is of sixth algebraic order,*
- *has the phase-lag and its first, second and third derivatives equal to zero on the first block layer (stages 1 and 2) of the hybrid method*
- *has the phase-lag and its first, second and third derivatives equal to zero on the second block level (stage 3) of the hybrid method*
- *has an interval of periodicity equals to: (0, 434.1806) in the case where the frequency of the scalar test equation for the phase-lag analysis is equal to the frequency of the scalar test equation for the stability analysis*

6 Numerical results

The investigation of the efficiency of the new produced hybrid type method is based on the approximate solution of the the radial time-independent Schrödinger equation (1).

The new obtained hybrid type method belonged to the frequency dependent methods. In this case and in order to be possible the application of the method to the radial Schrödinger equation, the value of parameter w must be defined. Based on (1), the parameter w is given by (for the case $l = 0$):

$$w = \sqrt{|V(r) - k^2|} = \sqrt{|V(r) - E|} \tag{44}$$

where $V(r)$ is the potential and E is the energy.

6.1 Woods–Saxon potential

The definition of the function of the potential is necessary for the approximate solution of the time-independent radial Schrödinger equation (1). For our numerical purposes, the well known Woods-Saxon potential is used. We can write this potential as

$$V(r) = \frac{u_0}{1+y} - \frac{u_0\, y}{a\,(1+y)^2} \tag{45}$$

with $y = \exp\left[\frac{r-X_0}{a}\right]$, $u_0 = -50$, $a = 0.6$, and $X_0 = 7.0$.

The behavior of Woods–Saxon potential is shown in Fig. 11.

In the literature several methods for the definition of the frequency w have been studied (see [25] and references therein). For our numerical tests we will use the following methodology (see for details [104]): Some critical points, which are obtained from the investigation of the appropriate potential, are determined.

Remark 6 The above mentioned methodology is well known applied to some potentials, such as the Woods–Saxon potential.

For the purpose of obtaining our numerical results, it is appropriate to choose v as follows (see for details [1,78]):

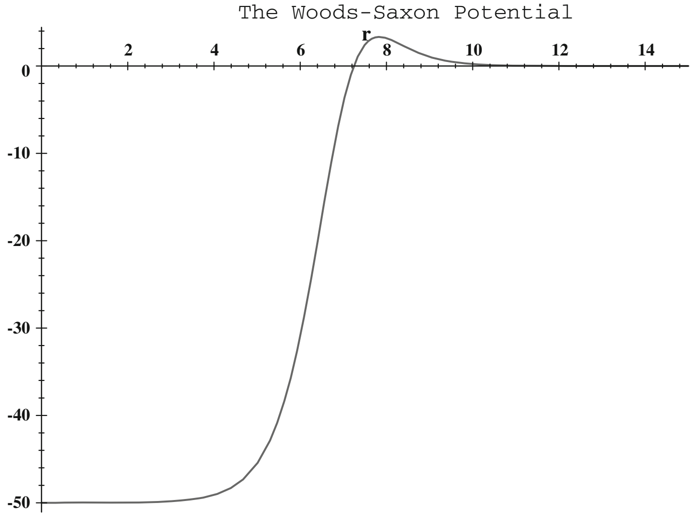


Fig. 11 The Woods–Saxon potential

$$w = \begin{cases} \sqrt{-50+E}, & \text{for } r \in [0, 6.5-2h], \\ \sqrt{-37.5+E}, & \text{for } r = 6.5-h \\ \sqrt{-25+E}, & \text{for } r = 6.5 \\ \sqrt{-12.5+E}, & \text{for } r = 6.5+h \\ \sqrt{E}, & \text{for } r \in [6.5+2h, 15] \end{cases} \tag{46}$$

For example, in the point of the integration region $r = 6.5 - h$, the value of w is equal to: $\sqrt{-37.5+E}$. So, $v = w\,h = \sqrt{-37.5+E}\,h$. In the point of the integration region $r = 6.5 - 3\,h$, the value of w is equal to: $\sqrt{-50+E}$, etc.

6.2 Radial Schrödinger equation: the resonance problem

Our numerical experiments consist the approximate solution of the radial time independent Schrödinger equation (1) using Woods–Saxon potential (45).

This problem has an infinite interval of integration which, in our case, is approximated by a finite one. For the experiments of this paper we take the integration interval $r \in [0, 15]$. We consider also a rather large domain of energies, i.e., $E \in [1, 1{,}000]$.

Remark 7 The potential decays faster than the term $\frac{l(l+1)}{r^2}$ in the case of positive energies, $E = k^2$

In the case described by the above mentioned remark the Schrödinger equation effectively reduces to:

$$p''(r) + \left(k^2 - \frac{l(l+1)}{r^2}\right) p(r) = 0 \tag{47}$$

for r greater than some value R.

The above equation has linearly independent solutions $krj_l(kr)$ and $krn_l(kr)$, where $j_l(kr)$ and $n_l(kr)$ are the spherical Bessel and Neumann functions respectively. Thus, the solution of Eq. (1) (when $r \to \infty$), has the asymptotic form

$$\begin{aligned} p(r) &\approx Akrj_l(kr) - Bkrn_l(kr) \\ &\approx AC\left[\sin\left(kr - \frac{l\pi}{2}\right) + \tan d_l \cos\left(kr - \frac{l\pi}{2}\right)\right] \end{aligned} \tag{48}$$

where δ_l is the phase shift that may be calculated from the formula

$$\tan \delta_l = \frac{p(r_2)S(r_1) - p(r_1)S(r_2)}{p(r_1)C(r_1) - p(r_2)C(r_2)} \tag{49}$$

for r_1 and r_2 distinct points in the asymptotic region (we choose r_1 as the right hand end point of the interval of integration and $r_2 = r_1 - h$) with $S(r) = krj_l(kr)$ and $C(r) = -krn_l(kr)$. In the initial-value problems (the radial Schrödinger equation is treated as an initial-value problem) we need $p_j,\ j = 0(1)3$ before starting a four-step method. From the initial condition, we obtain p_0. The values $p_i,\ i = 1(1)3$ are obtained by using high order Runge–Kutta–Nyström methods(see [111,112]). With these starting values, we evaluate at r_2 of the asymptotic region the phase shift δ_l.

For positive energies, we have the so-called resonance problem. This problem consists either of finding the phase-shift δ_l or finding those E, for $E \in [1, 1{,}000]$, at which $\delta_l = \frac{\pi}{2}$. We actually solve the latter problem, known as *the resonance problem*.

The boundary conditions for this problem are:

$$p(0) = 0, \ \ p(r) = \cos\left(\sqrt{E}r\right) \ \text{for large } r. \tag{50}$$

We compute the approximate positive eigenenergies of the Woods-Saxon resonance problem using:

- The eighth order multi-step method developed by Quinlan and Tremaine [19], which is indicated as *Method QT8*.
- The tenth order multi-step method developed by Quinlan and Tremaine [19], which is indicated as *Method QT10*.
- The twelfth order multi-step method developed by Quinlan and Tremaine [19], which is indicated as *Method QT12*.
- The fourth algebraic order method of Chawla and Rao with minimal phase-lag [24], which is indicated as *Method MCR4*
- The exponentially fitted method of Raptis and Allison [79], which is indicated as *Method MRA*
- The hybrid sixth algebraic order method developed by Chawla and Rao with minimal phase-lag [23], which is indicated as *Method MCR6*
- The classical form of the sixth algebraic order four-step method developed in Sect. 4, which is indicated as *Method NMCL*.[2]

[2] With the term classical we mean the method of Sect. 4 with constant coefficients.

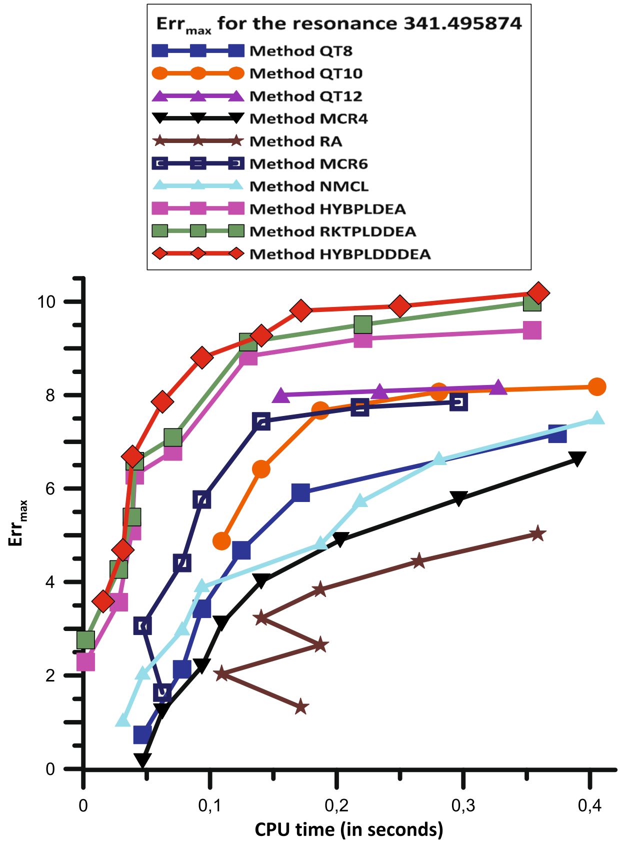


Fig. 12 Accuracy (digits) for several values of CPU Time (in seconds) for the eigenvalue $E_2 = 341.495874$. The nonexistence of a value of accuracy (digits) indicates that for this value of CPU, accuracy (digits) is <0

- The four-step method of sixth algebraic order with vanished phase-lag and its first derivative in each level (obtained in [40]), which is indicated as *Method HYBPLDEA*
- The four-step method of sixth algebraic order with vanished phase-lag and its first and second derivatives in each level (obtained in [41]), which is indicated as *Method RKTPLDDEA*
- The four-step method of sixth algebraic order with vanished phase-lag and its first, second and third derivatives in each level (obtained in Sect. 3), which is indicated as *Method HYBPLDDDEA*

The numerically calculated eigenenergies are compared with reference values.[3] In Figs. 12 and 13, we present the maximum absolute error $Err_{max} = |log_{10}(Err)|$ where

[3] The reference values are computed using the well known two-step method of Chawla and Rao [23] with small step size for the integration

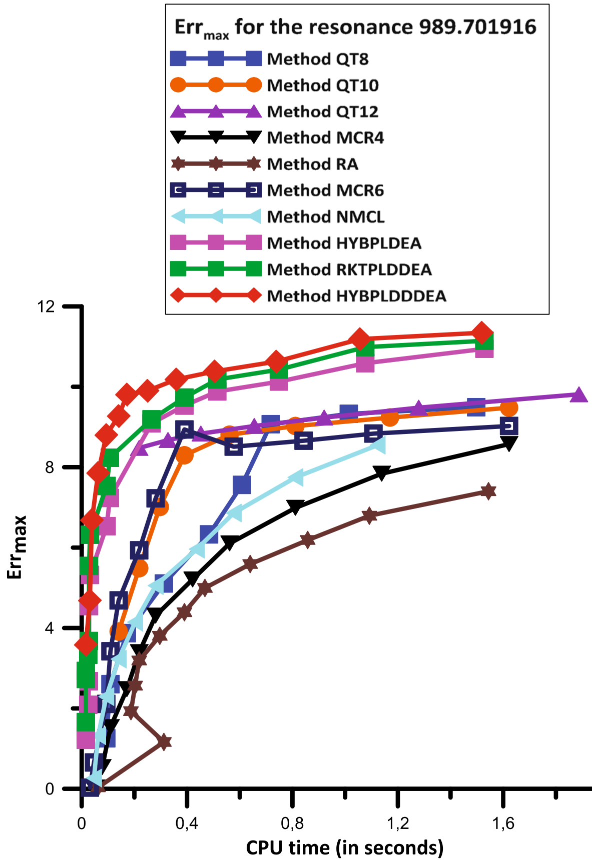


Fig. 13 Accuracy (digits) for several values of CPU Time (in seconds) for the eigenvalue $E_3 = 989.701916$. The nonexistence of a value of accuracy (digits) indicates that for this value of CPU, accuracy (digits) is <0

$$Err = |E_{calculated} - E_{accurate}| \tag{51}$$

of the eigenenergies $E_2 = 341.495874$ and $E_3 = 989.701916$ respectively, for several values of CPU time (in seconds). We note that the CPU time (in seconds) counts the computational cost for each method.

7 Conclusions

In this paper, we presented a new methodology for the development of four-step hybrid type methods of sixth algebraic order with vanished phase-lag and its derivatives. This new methodology is based on the vanishing of the phase-lag and its derivatives in each level of the hybrid method. We have also investigated the influencing of the vanishing of the phase-lag and its first derivative on the efficiency of the above mentioned methods for the numerical solution of the radial Schrödinger equation and related problems.

Based on the the above, a two-stage four-step sixth algebraic order methods with vanished phase-lag and its first derivative in each level was obtained. This new method is very efficient on any problem with oscillating solutions or problems with solutions contain the functions cos and sin or any combination of them.

From the results presented above, we can make the following remarks:

1. The classical form of the sixth algebraic order four-step method developed in Sect. 4, which is indicated as *Method NMCL* is more efficient than the fourth algebraic order method of Chawla and Rao with minimal phase-lag [24], which is indicated as *Method MCR4*. Both the above mentioned methods are more efficient than the exponentially fitted method of Raptis and Allison [79], which is indicated as *Method MRA*.
2. The tenth algebraic order multistep method developed by Quinlan and Tremaine [19], which is indicated as *Method QT10* is more efficient than the fourth algebraic order method of Chawla and Rao with minimal phase-lag [24], which is indicated as *Method MCR4*. The *Method QT10* is also more efficient than the eighth order multi-step method developed by Quinlan and Tremaine [19], which is indicated as *Method QT8*. Finally, the *Method QT10* is more efficient than the hybrid sixth algebraic order method developed by Chawla and Rao with minimal phase-lag [23], which is indicated as *Method MCR6* for large CPU time and less efficient than the *Method MCR6* for small CPU time.
3. The twelfth algebraic order multistep method developed by Quinlan and Tremaine [19], which is indicated as *Method QT12* is more efficient than the tenth order multistep method developed by Quinlan and Tremaine [19], which is indicated as *Method QT10*
4. The hybrid four-step two-stage sixth algebraic order method with vanished phase-lag and its first derivative in each level of the method (obtained in [40]), which is indicated as *Method HYBPLDEA* is the more efficient than all the above mentioned methods.
5. The four-step method of sixth algebraic order with vanished phase-lag and its first and second derivatives in each level (obtained in [41]), which is indicated as *Method RKTPLDDEA*, is more efficient than all the above mentioned methods.
6. The four-step method of sixth algebraic order with vanished phase-lag and its first, second and third derivatives in each level (obtained in Sect. 3), which is indicated as *Method HYBPLDDDEA*, is the most efficient one.

All computations were carried out on a IBM PC-AT compatible 80486 using double precision arithmetic with 16 significant digits accuracy (IEEE standard).

References

1. L. Gr. Ixaru, M. Micu, *Topics in Theoretical Physics* (Central Institute of Physics, Bucharest, 1978)
2. L.D. Landau, F.M. Lifshitz, *Quantum Mechanics* (Pergamon, New York, 1965)
3. I. Prigogine, S. Rice (eds), *Advances in Chemical Physics, Vol. 93: New Methods in Computational Quantum Mechanics* (Wiley, London, 1997)
4. G. Herzberg, *Spectra of Diatomic Molecules* (Van Nostrand, Toronto, 1950)
5. T.E. Simos, J. Vigo-Aguiar, A modified phase-fitted Runge–Kutta method for the numerical solution of the Schrödinger equation. J. Math. Chem. **30**(1), 121–131 (2001)

6. K. Tselios, T.E. Simos, Runge–Kutta methods with minimal dispersion and dissipation for problems arising from computational acoustics. J. Comput. Appl. Math. **175**(1), 173–181 (2005)
7. Z.A. Anastassi, T.E. Simos, An optimized Runge–Kutta method for the solution of orbital problems. J. Comput. Appl. Math. **175**(1), 1–9 (2005)
8. D.F. Papadopoulos, T.E. Simos, A new methodology for the construction of optimized Runge–Kutta–Nyström methods. Int. J. Mod. Phys. C **22**(6), 623–634 (2011)
9. Dimitris F. Papadopoulos, T.E. Simos, A modified Runge–Kutta–Nyström method by using phase lag properties for the numerical solution of orbital problems. Appl. Math. Inf. Sci. **7**(2), 433–437 (2013)
10. Th Monovasilis, Z. Kalogiratou, T.E. Simos, Exponentially fitted symplectic Runge–Kutta–Nyström methods. Appl. Math. Inf. Sci. **7**(1), 81–85 (2013)
11. A.A. Kosti, Z.A. Anastassi, T.E. Simos, An optimized explicit Runge-Kutta method with increased phase-lag order for the numerical solution of the Schrödinger equation and related problems. J. Math. Chem. **47**(1), 315–330 (2010)
12. Z. Kalogiratou, T.E. Simos, Construction of trigonometrically and exponentially fitted Runge–Kutta–Nyström methods for the numerical solution of the Schrödinger equation and related problems a method of 8th algebraic order. J. Math. Chem. **31**(2), 211–232 (2002)
13. T.E. Simos, A fourth algebraic order exponentially-fitted Runge–Kutta method for the numerical solution of the Schrödinger equation. IMA J. Numer. Anal. **21**(4), 919–931 (2001)
14. T.E. Simos, Exponentially-fitted Runge–Kutta–Nyström method for the numerical solution of initial-value problems with oscillating solutions. Appl. Math. Lett. **15**(2), 217–225 (2002)
15. Ch. Tsitouras, T.E. Simos, Optimized Runge–Kutta pairs for problems with oscillating solutions. J. Comput. Appl. Math. **147**(2), 397–409 (2002)
16. Z.A. Anastassi, T.E. Simos, Trigonometrically fitted Runge–Kutta methods for the numerical solution of the Schrödinger equation. J. Math. Chem. **37**(3), 281–293 (2005)
17. Z.A. Anastassi, T.E. Simos, A family of exponentially-fitted Runge–Kutta methods with exponential order up to three for the numerical solution of the Schrödinger equation. J. Math. Chem. **41**(1), 79–100 (2007)
18. J.D. Lambert, I.A. Watson, Symmetric multistep methods for periodic initial values problems. J. Inst. Math. Appl. **18**, 189–202 (1976)
19. G.D. Quinlan, S. Tremaine, Symmetric multistep methods for the numerical integration of planetary orbits. Astron. J. **100**, 1694–1700 (1990)
20. C. Tsitouras, I.T. Famelis, T.E. Simos, On modified Runge–Kutta trees and methods. Comput. Math. Appl. **62**(4), 2101–2111 (2011)
21. http://burtleburtle.net/bob/math/multistep.html
22. G. Avdelas, A. Konguetsof, T.E. Simos, A generator and an optimized generator of high-order hybrid explicit methods for the numerical solution of the Schrödinger equation. Part 1. Development of the basic method. J. Math. Chem. **29**(4), 281–291 (2001)
23. M.M. Chawla, P.S. Rao, An explicit sixth-order method with phase-lag of order eight for $y'' = f(t, y)$. J. Comput. Appl. Math. **17**, 363–368 (1987)
24. M.M. Chawla, P.S. Rao, An Noumerov-typ method with minimal phase-lag for the integration of second order periodic initial-value problems II explicit method. J. Comput. Appl. Math. **15**, 329–337 (1986)
25. T.E. Simos, P.S. Williams, A finite difference method for the numerical solution of the Schrödinger equation. J. Comput. Appl. Math. **79**, 189–205 (1997)
26. G. Avdelas, A. Konguetsof, T.E. Simos, A generator and an optimized generator of high-order hybrid explicit methods for the numerical solution of the Schrödinger equation. Part 2. Development of the generator; optimization of the generator and numerical results. J. Math. Chem. **29**(4), 293–305 (2001)
27. T.E. Simos, J. Vigo-Aguiar, Symmetric eighth algebraic order methods with minimal phase-lag for the numerical solution of the Schrödinger equation. J. Math. Chem. **31**(2), 135–144 (2002)
28. A. Konguetsof, T.E. Simos, A generator of hybrid symmetric four-step methods for the numerical solution of the Schrödinger equation. J. Comput. Appl. Math. **158**(1), 93–106 (2003)
29. T.E. Simos, I.T. Famelis, C. Tsitouras, Zero dissipative, explicit numerov-type methods for second order IVPs with oscillating solutions. Numer. Algorithms **34**(1), 27–40 (2003)
30. D.P. Sakas, T.E. Simos, Multiderivative methods of eighth algrebraic order with minimal phase-lag for the numerical solution of the radial Schrödinger equation. J. Comput. Appl. Math. **175**(1), 161–172 (2005)

31. T.E. Simos, Optimizing a class of linear multi-step methods for the approximate solution of the radial Schrödinger equation and related problems with respect to phase-lag. Cent. Eur. J. Phys. **9**(6), 1518–1535 (2011)
32. D.P. Sakas, T.E. Simos, A family of multiderivative methods for the numerical solution of the Schrödinger equation. J. Math. Chem. **37**(3), 317–331 (2005)
33. H. Van de Vyver, Phase-fitted and amplification-fitted two-step hybrid methods for $y'' = f(x, y)$. J. Comput. Appl. Math. **209**(1), 33–53 (2007)
34. Hans Van de Vyver, An explicit Numerov-type method for second-order differential equations with oscillating solutions. Comput. Math. Appl. **53**, 1339–1348 (2007)
35. T.E. Simos, A new Numerov-type method for the numerical solution of the Schrödinger equation. J. Math. Chem. **46**(3), 981–1007 (2009)
36. G.A. Panopoulos, Z.A. Anastassi, T.E. Simos, A new symmetric eight-step predictor-corrector method for the numerical solution of the radial Schrödinger equation and related orbital problems. Int. J. Mod. Phys. C **22**(2), 133–153 (2011)
37. G.A. Panopoulos, Z.A. Anastassi, T.E. Simos, A symmetric eight-step predictor-corrector method for the numerical solution of the radial Schrödinger equation and related IVPs with oscillating solutions. Comput. Phys. Commun. **182**(8), 1626–1637 (2011)
38. T.E. Simos, Optimizing a hybrid two-step method for the numerical solution of the Schrödinger equation and related problems with respect to phase-lag. J. Appl. Math. Article ID 420387, Volume 2012 (2012).
39. T.E. Simos, A two-step method with vanished phase-lag and its first two derivatives for the numerical solution of the Schrödinger equation. J. Math. Chem. **49**(10), 2486–2518 (2011)
40. I. Alolyan, T.E. Simos, A new four-step hybrid type method with vanished phase-lag and its first derivatives for each level for the approximate integration of the Schrödinger equation. J. Math. Chem. **51**, 2542–2571 (2013)
41. I. Alolyan, T.E. Simos, A Runge–Kutta type four-step method with vanished phase-lag and its first and second derivatives for each level for the numerical integration of the Schrödinger equation. J. Math. Chem. **52**, 917–947 (2014)
42. Z.A. Anastassi, T.E. Simos, A parametric symmetric linear four-step method for the efficient integration of the Schrödinger equation and related oscillatory problems. J. Comput. Appl. Math. **236**(16), 3880–3889 (2012)
43. G.A. Panopoulos, T.E. Simos, An optimized symmetric 8-step semi-embedded predictor-corrector method for IVPs with oscillating solutions. Appl. Math. Inf. Sci. **7**(1), 73–80 (2013)
44. G.A. Panopoulos, Z.A. Anastassi, T.E. Simos, A new eight-step symmetric embedded predictor-corrector method (EPCM) for orbital problems and related IVPs with oscillatory solutions. Astron. J. **145**(3) Article Number: 75 (2013). doi:10.1088/0004-6256/145/3/75
45. T.E. Simos, New high order multiderivative explicit four-step methods with vanished phase-lag and its derivatives for the approximate solution of the Schrödinger equation. Construction and theoretical analysis. J. Math. Chem. **51**(1), 194–226 (2013)
46. T.E. Simos, On the explicit four-step methods with vanished phase-lag and its first derivative. Appl. Math. Inf. Sci. **8**(2), 447–458 (2014)
47. G.A. Panopoulos, T.E. Simos, A new optimized symmetric embedded predictor-corrector method (EPCM) for initial-value problems with oscillatory solutions. Appl. Math. Inf. Sci. **8**(2), 703–713 (2014)
48. A. Konguetsof, A new two-step hybrid method for the numerical solution of the Schrödinger equation. J. Math. Chem. **47**(2), 871–890 (2010)
49. K. Tselios, T.E. Simos, Symplectic methods for the numerical solution of the radial Shrödinger equation. J. Math. Chem. **34**(1–2), 83–94 (2003)
50. K. Tselios, T.E. Simos, Symplectic methods of fifth order for the numerical solution of the radial Shrodinger equation. J. Math. Chem. **35**(1), 55–63 (2004)
51. T. Monovasilis, T.E. Simos, New second-order exponentially and trigonometrically fitted symplectic integrators for the numerical solution of the time-independent Schrödinger equation. J. Math. Chem. **42**(3), 535–545 (2007)
52. T. Monovasilis, Z. Kalogiratou, T.E. Simos, Exponentially fitted symplectic methods for the numerical integration of the Schrödinger equation. J. Math. Chem. **37**(3), 263–270 (2005)

53. T. Monovasilis, Z. Kalogiratou, T.E. Simos, Trigonometrically fitted and exponentially fitted symplectic methods for the numerical integration of the Schrödinger equation. J. Math. Chem. **40**(3), 257–267 (2006)
54. Z. Kalogiratou, T. Monovasilis, T.E. Simos, Symplectic integrators for the numerical solution of the Schrödinger equation. J. Comput. Appl. Math. **158**(1), 83–92 (2003)
55. T.E. Simos, Closed Newton–Cotes trigonometrically-fitted formulae of high-order for long-time integration of orbital problems. Appl. Math. Lett. **22**(10), 1616–1621 (2009)
56. Z. Kalogiratou, T.E. Simos, Newton–Cotes formulae for long-time integration. J. Comput. Appl. Math. **158**(1), 75–82 (2003)
57. T.E. Simos, High order closed Newton-Cotes trigonometrically-fitted formulae for the numerical solution of the Schrödinger equation. Appl. Math. Comput. **209**(1), 137–151 (2009)
58. T.E. Simos, Closed Newton–Cotes trigonometrically-fitted formulae for the solution of the Schrödinger equation. MATCH Commun. Math. Comput. Chem. **60**(3), 787–801 (2008)
59. T.E. Simos, Closed Newton–Cotes trigonometrically-fitted formulae of high order for the numerical integration of the Schrödinger equation. J. Math. Chem. **44**(2), 483–499 (2008)
60. T.E. Simos, High-order closed Newton–Cotes trigonometrically-fitted formulae for long-time integration of orbital problems. Comput. Phys. Commun. **178**(3), 199–207 (2008)
61. T.E. Simos, Closed Newton–Cotes trigonometrically-fitted formulae for numerical integration of the Schrödinger equation. Comput. Lett. **3**(1), 45–57 (2007)
62. T.E. Simos, Closed Newton–Cotes trigonometrically-fitted formulae for long-time integration of orbital problems. RevMexAA **42**(2), 167–177 (2006)
63. T.E. Simos, Closed Newton–Cotes trigonometrically-fitted formulae for long-time integration. Int. J. Mod. Phys. C **14**(8), 1061–1074 (2003)
64. T.E. Simos, New closed Newton–Cotes type formulae as multilayer symplectic integrators. J. Chem. Phys. **133**(10) Article Number: 104108 (2010)
65. T.E. Simos, New stable closed Newton–Cotes trigonometrically fitted formulae for long-time integration. Abstr. Appl. Anal. Article Number: 182536 (2012). doi:10.1155/2012/182536
66. T.E. Simos, High order closed Newton–Cotes exponentially and trigonometrically fitted formulae as multilayer symplectic integrators and their application to the radial Schrödinger equation. J. Math. Chem. **50**(5), 1224–1261 (2012)
67. T.E. Simos, Accurately Closed Newton–Cotes trigonometrically-fitted formulae for the numerical solution of the Schrödinger equation. Int. J. Mod. Phys. C **24**(3) (2013). doi:10.1142/S0129183113500149
68. T.E. Simos, New open modified Newton–Cotes type formulae as multilayer symplectic integrators. Appl. Math. Model. **37**(4), 1983–1991 (2013)
69. G. Vanden Berghe, M. Van Daele, Exponentially fitted open Newton–Cotes differential methods as multilayer symplectic integrators. J. Chem. Phys. **132**, 204107 (2010)
70. Z. Kalogiratou, T. Monovasilis, T.E. Simos, A fifth-order symplectic trigonometrically fitted partitioned Runge-Kutta method, in *International Conference on Numerical Analysis and Applied Mathematics, SEP 16–20, 2007 Corfu, GREECE, Numerical Analysis and Applied Mathematics, AIP Conference Proceedings***936**, 313–317 (2007).
71. T. Monovasilis, Z. Kalogiratou, T.E. Simos, Families of third and fourth algebraic order trigonometrically fitted symplectic methods for the numerical integration of Hamiltonian systems. Comput. Phys. Commun. **177**(10), 757–763 (2007)
72. T. Monovasilis, T.E. Simos, Symplectic methods for the numerical integration of the Schrödinger equation. Comput. Mater. Sci. **38**(3), 526–532 (2007)
73. T. Monovasilis, Z. Kalogiratou, T.E. Simos, Computation of the eigenvalues of the Schrödinger equation by symplectic and trigonometrically fitted symplectic partitioned Runge-Kutta methods. Phys. Lett. A **372**(5), 569–573 (2008)
74. Z. Kalogiratou, Th Monovasilis, T.E. Simos, New modified Runge–Kutta–Nyström methods for the numerical integration of the Schrödinger equation. Comput. Math. Appl. **60**(6), 1639–1647 (2010)
75. Th Monovasilis, Z. Kalogiratou, T.E. Simos, Two new phase-fitted symplectic partitioned Runge–Kutta methods. Int. J. Mod. Phys. C **22**(12), 1343–1355 (2011)
76. K. Tselios, T.E. Simos, Optimized fifth order symplectic integrators for orbital problems. Rev. Mex. Astron. Astrofis. **49**(1), 11–24 (2013)
77. T. Monovasilis, Z. Kalogiratou, T.E. Simos, Symplectic partitioned Runge–Kutta methods with minimal phase-lag. Comput. Phys. Commun. **181**(7), 1251–1254 (2010)

78. L.Gr Ixaru, M. Rizea, A Numerov-like scheme for the numerical solution of the Schrödinger equation in the deep continuum spectrum of energies. Comput. Phys. Commun. **19**, 23–27 (1980)
79. A.D. Raptis, A.C. Allison, Exponential-fitting methods for the numerical solution of the Schrödinger equation. Comput. Phys. Commun. **14**, 1–5 (1978)
80. J. Vigo-Aguiar, T.E. Simos, Family of twelve steps exponential fitting symmetric multistep methods for the numerical solution of the Schrödinger equation. J. Math. Chem. **32**(3), 257–270 (2002)
81. G. Psihoyios, T.E. Simos, Trigonometrically fitted predictor-corrector methods for IVPs with oscillating solutions. J. Comput. Appl. Math. **158**(1), 135–144 (2003)
82. G. Psihoyios, T.E. Simos, A fourth algebraic order trigonometrically fitted predictor-corrector scheme for IVPs with oscillating solutions. J. Comput. Appl. Math. **175**(1), 137–147 (2005)
83. T.E. Simos, Dissipative trigonometrically-fitted methods for linear second-order IVPs with oscillating solution. Appl. Math. Lett. **17**(5), 601–607 (2004)
84. T.E. Simos, Exponentially and trigonometrically fitted methods for the solution of the Schrödinger equation. Acta Appl. Math. **110**(3), 1331–1352 (2010)
85. G. Avdelas, E. Kefalidis, T.E. Simos, New P-stable eighth algebraic order exponentially-fitted methods for the numerical integration of the Schrödinger equation. J. Math. Chem. **31**(4), 371–404 (2002)
86. T.E. Simos, A family of trigonometrically-fitted symmetric methods for the efficient solution of the Schrödinger equation and related problems J. Math. Chem. **34**(1–2), 39–58 (2003)
87. T.E. Simos, Exponentially-fitted multiderivative methods for the numerical solution of the Schrödinger equation. J. Math. Chem. **36**(1), 13–27 (2004)
88. T.E. Simos, A four-step exponentially fitted method for the numerical solution of the Schrödinger equation. J. Math. Chem. **40**(3), 305–318 (2006)
89. H. Van de Vyver, A trigonometrically fitted explicit hybrid method for the numerical integration of orbital problems. Appl. Math. Comput. **189**(1), 178–185 (2007)
90. T.E. Simos, A family of four-step trigonometrically-fitted methods and its application to the Schrodinger equation. J. Math. Chem. **44**(2), 447–466 (2009)
91. Z.A. Anastassi, T.E. Simos, A family of two-stage two-step methods for the numerical integration of the Schrödinger equation and related IVPs with oscillating solution. J. Math. Chem. **45**(4), 1102–1129 (2009)
92. G. Psihoyios, T.E. Simos, Sixth algebraic order trigonometrically fitted predictor-corrector methods for the numerical solution of the radial Schrödinger equation. J. Math. Chem. **37**(3), 295–316 (2005)
93. G. Psihoyios, T.E. Simos, The numerical solution of the radial Schrödinger equation via a trigonometrically fitted family of seventh algebraic order predictor-corrector methods. J. Math. Chem. **40**(3), 269–293 (2006)
94. Z. Wang, P-stable linear symmetric multistep methods for periodic initial-value problems. Comput. Phys. Commun. **171**(3), 162–174 (2005)
95. T.E. Simos, A new explicit Bessel and Neumann fitted eighth algebraic order method for the numerical solution of the Schrödinger equation. J. Math. Chem. **27**(4), 343–356 (2000)
96. Z.A. Anastassi, T.E. Simos, A family of two-stage two-step methods for the numerical integration of the Schrödinger equation and related IVPs with oscillating solution. J. Math. Chem. **45**(4), 1102–1129 (2009)
97. C. Tang, W. Wang, H. Yan, Z. Chen, High-order predictorcorrector of exponential fitting for the N-body problems. J. Comput. Phys. **214**(2), 505–520 (2006)
98. G.A. Panopoulos, Z.A. Anastassi, T.E. Simos, Two optimized symmetric eight-step implicit methods for initial-value problems with oscillating solutions. J. Math. Chem. **46**(2), 604–620 (2009)
99. S. Stavroyiannis, T.E. Simos, Optimization as a function of the phase-lag order of nonlinear explicit two-step P-stable method for linear periodic IVPs. Appl. Numer. Math. **59**(10), 2467–2474 (2009)
100. S. Stavroyiannis, T.E. Simos, A nonlinear explicit two-step fourth algebraic order method of order infinity for linear periodic initial value problems. Comput. Phys. Commun. **181**(8), 1362–1368 (2010)
101. Z.A. Anastassi, T.E. Simos, Numerical multistep methods for the efficient solution of quantum mechanics and related problems. Phys. Rep. **482**, 1–240 (2009)
102. R. Vujasin, M. Sencanski, J. Radic-Peric, M. Peric, A comparison of various variational approaches for solving the one-dimensional vibrational Schrödinger equation. MATCH Commun. Math. Comput. Chem. **63**(2), 363–378 (2010)
103. T.E. Simos, P.S. Williams, On finite difference methods for the solution of the Schrödinger equation. Comput. Chem. **23**, 513–554 (1999)

104. L. Gr. Ixaru, M. Rizea, Comparison of some four-step methods for the numerical solution of the Schrödinger equation. Comput. Phys. Commun. **38**(3), 329–337 (1985)
105. J. Vigo-Aguiar, T.E. Simos, Review of multistep methods for the numerical solution of the radial Schrödinger equation. Int. J. Quantum Chem. **103**(3), 278–290 (2005)
106. T.E. Simos, G. Psihoyios, Special issue: The international conference on computational methods in sciences and engineering 2004—preface. J. Comput. Appl. Math. **191**(2), 165 (2006)
107. T.E. Simos and G. Psihoyios, Special issue: Selected papers of the international conference on computational methods in sciences and engineering (ICCMSE 2003) Kastoria, Greece, 12–16 September 2003—preface. J. Comput. Appl. Math. **175**(1), IX–IX (2005)
108. T.E. Simos, J. Vigo-Aguiar, Special Issue: Selected papers from the conference on computational and mathematical methods for science and engineering (CMMSE-2002)—Alicante University, Spain, 20–25 September 2002—preface. J. Comput. Appl. Math. **158**(1), IX–IX (2003)
109. T.E. Simos, Preface for the special issue on the International Conference of Computational Methods in Sciences and Engineering 2005 (ICCMSE 2005). Appl. Math. Comput. **209**(1), 1 (2009)
110. T.E. Simos, I. Gutman, Papers presented on the international conference on computational methods in sciences and engineering (Castoria, Greece, September 12–16, 2003), MATCH Commun. Math. Comput. Chem. **53**(2), A3–A4 (2005)
111. J.R. Dormand, M.E.A. El-Mikkawy, P.J. Prince, Families of Runge–Kutta–Nyström formulae. IMA J. Numer. Anal. **7**, 235–250 (1987)
112. J.R. Dormand, P.J. Prince, A family of embedded Runge–Kutta formulae. J. Comput. Appl. Math. **6**, 19–26 (1980)